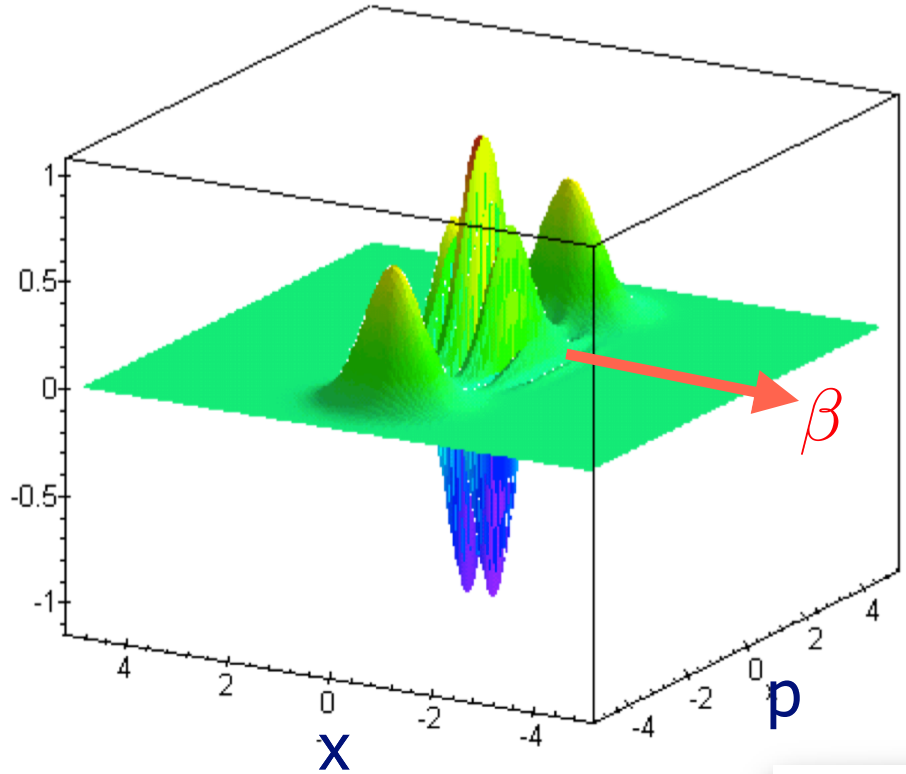
Lecture 3

EXPERIMENT:
Measuring sub-Planck state displacements in phase space



$$\boxed{\beta \approx \frac{1}{|\alpha|} \Rightarrow \text{orthogonality}}$$

$$|\mathrm{cat}\rangle = \left(|\alpha\rangle + |-\alpha\rangle\right)/\sqrt{2}$$

PHYSICAL REVIEW A **73**, 023803 (2006)

Sub-Planck phase-space structures and Heisenberg-limited measurements

F. Toscano,[1] D. A. R. Dalvit,[2] L. Davidovich,[1] and W. H. Zurek[2]

PHYSICAL REVIEW A **94**, 022313 (2016)

Measurement of a microwave field amplitude beyond the standard quantum limit

M. Penasa,[1] S. Gerlich,[1] T. Rybarczyk,[1] V. Métillon,[1] M. Brune,[1] J. M. Raimond,[1] S. Haroche,[1] L. Davidovich,[2] and I. Dotsenko[1,*]

Looking for a classical-like distribution in phase space

We look for a distribution in phase space with the following property:

$$\int dp W(q,p) = \langle q|\hat{\rho}|q\rangle, \ \int dq W(q,p) = \langle p|\hat{\rho}|p\rangle$$

Pure state:

$$\langle q|\hat{\rho}|q\rangle = |\psi(q)|^2, \ \langle p|\hat{\rho}|p\rangle = |\tilde{\psi}(p)|^2$$

Property should be valid with rotated axes:

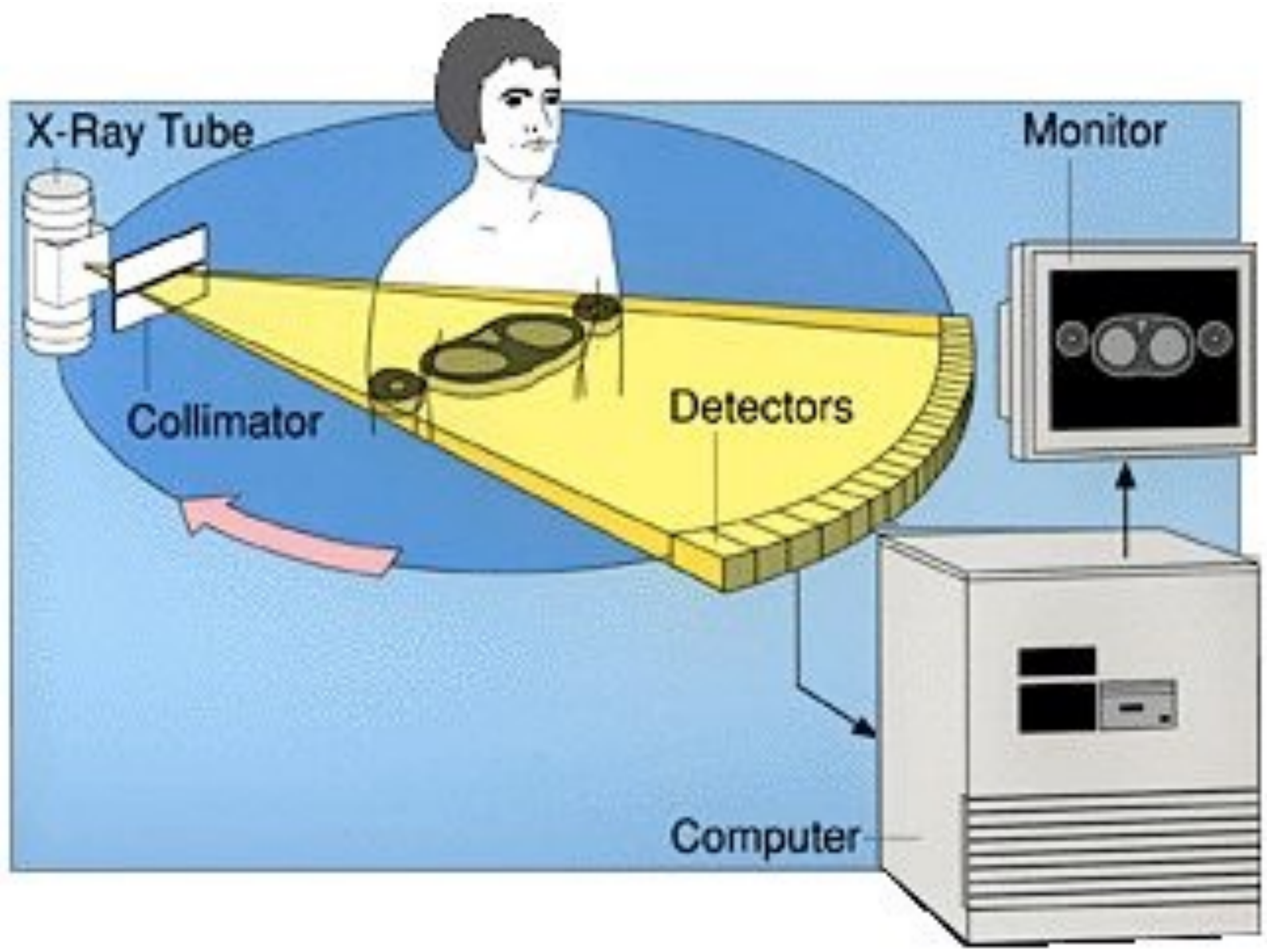
$$\int W(q_\theta \cos\theta - p_\theta \sin\theta, q_\theta \sin\theta + p_\theta \cos\theta) dp_\theta$$

$$= P(q_\theta) = \langle q|\hat{U}^\dagger(\theta)\hat{\rho}\hat{U}(\theta)|q\rangle$$

RADON TRANSFORM (1917)

$P(q_\theta)$ determines uniquely $W(q,p)$! $\rightarrow$ inverse Radon transform

$\rightarrow$ tomography



Cormack and Hounsfield: Nobel Prize in Medicine (1979)

Quantum mechanics: $P(q_\theta)$ $\Rightarrow$ Wigner distribution

(Bertrand and Bertrand, 1987)

Wigner distribution

Wigner, 1932: Quantum corrections to classical statistical mechanics

$$\hat{x}|x\rangle = x|x\rangle$$

$$W(x,p) = \frac{1}{\pi\hbar}\int \langle x+x'|\hat{\rho}|x-x'\rangle e^{-2ipx'/\hbar}dx'$$

Moyal, 1949: Average of operators in symmetric form

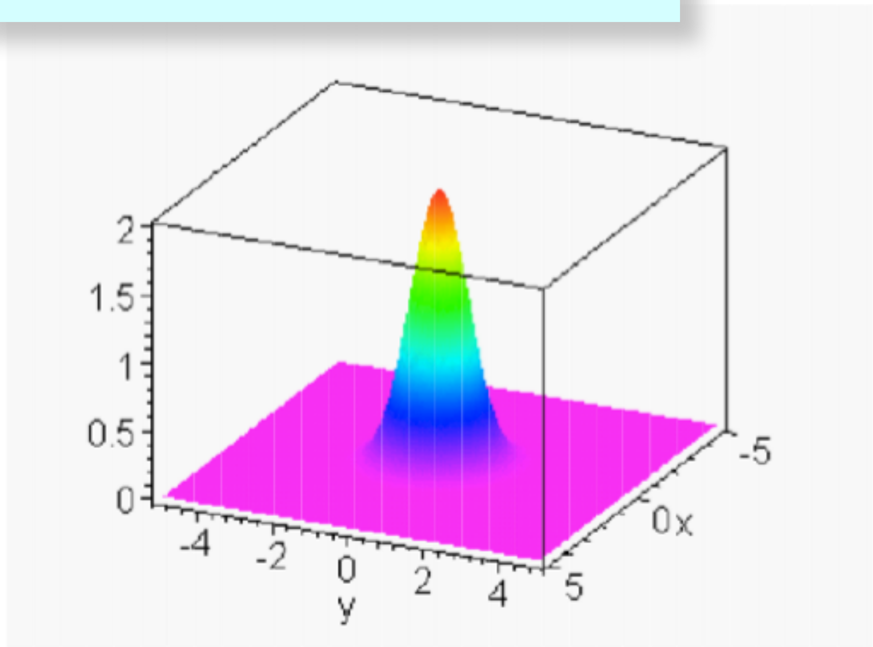
$$Tr\left[\hat{\rho}\left(\hat{x}\hat{p}+\hat{p}\hat{x}\right)/2\right] = \int dxdpW(x,p)xp$$

Density matrix from W:

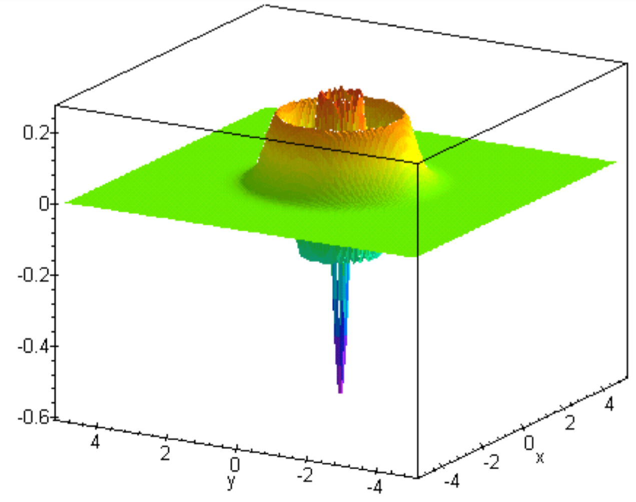
$$\langle x+x'|\hat{\rho}|x-x'\rangle = \int W(x,p)e^{2ipx'/\hbar}dp/\hbar$$

Examples of Wigner distributions for harmonic oscillator

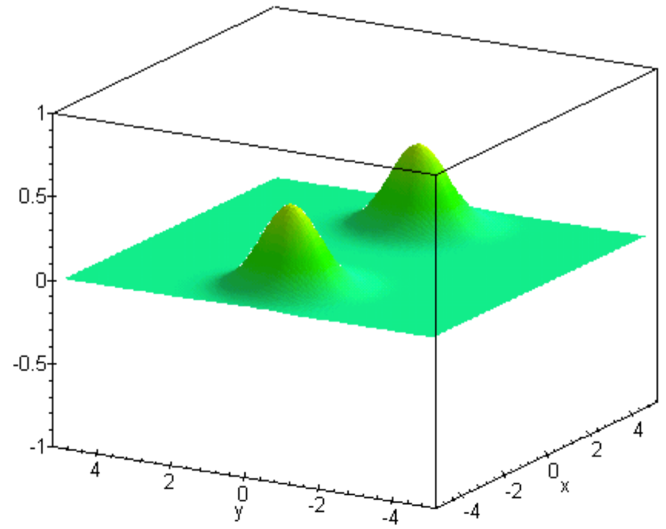
Ground state



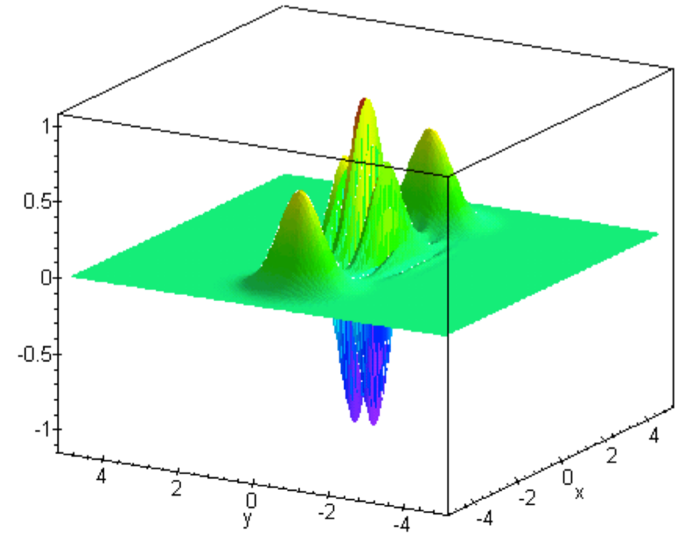
Fock state with n=3



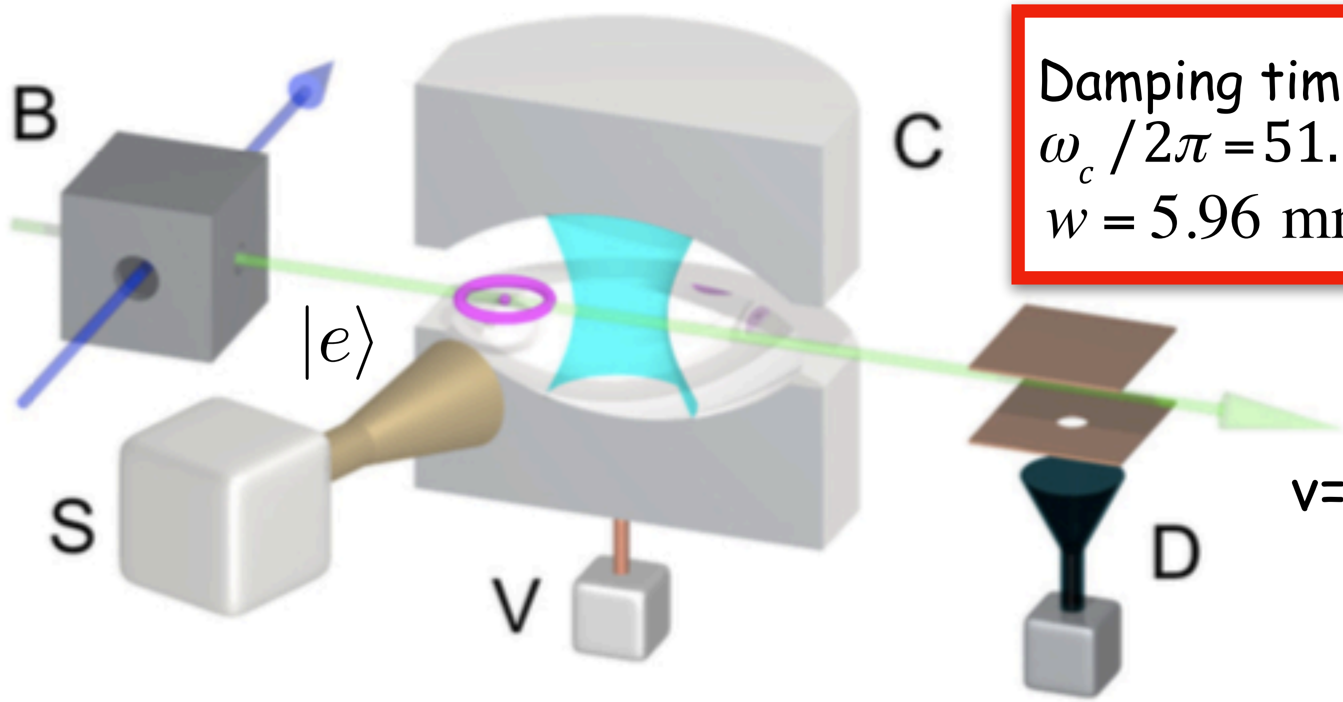
Mixed state $(|\alpha\rangle\langle\alpha|+|-\alpha\rangle\langle-\alpha|)/2$



Superposition $\propto\ |\alpha\rangle+|-\alpha\rangle$



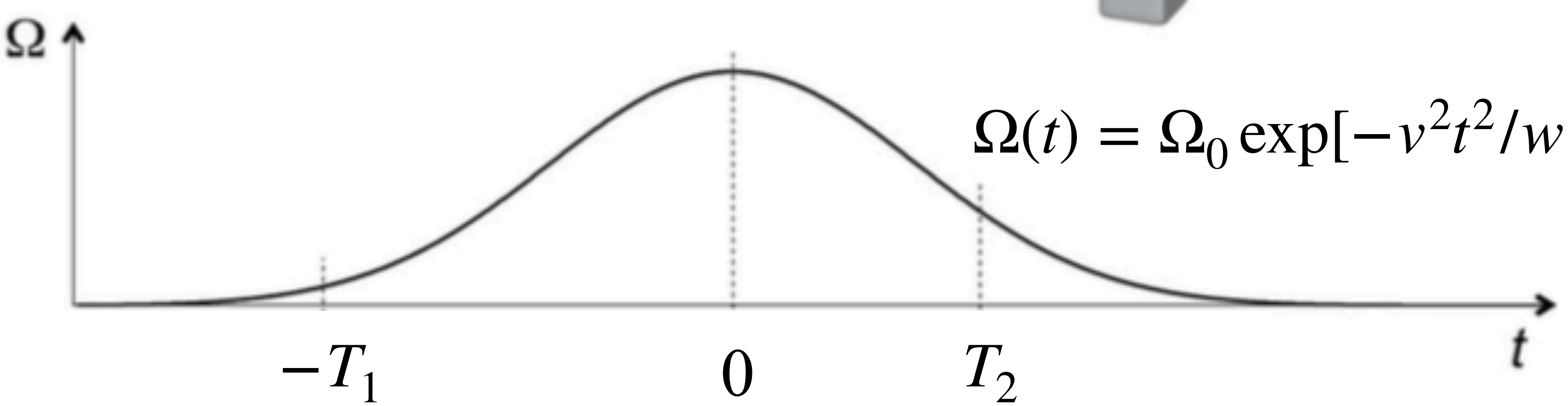
Experimental procedure



Damping time 65 ms
$\omega_c / 2\pi = 51.1$ GHz
$w = 5.96$ mm

Coherent state
with 12.7 photons
$\{|g\rangle, |e\rangle\} \to n = 50, 51$
$\Omega_0 / 2\pi = 46$ kHz

v=250 m/s



$$\Omega(t) = \Omega_0 \exp[-v^2 t^2 / w^2]$$

Temporal variation of the atom-cavity coupling

$T_{\max} \simeq 42\ \mu s$

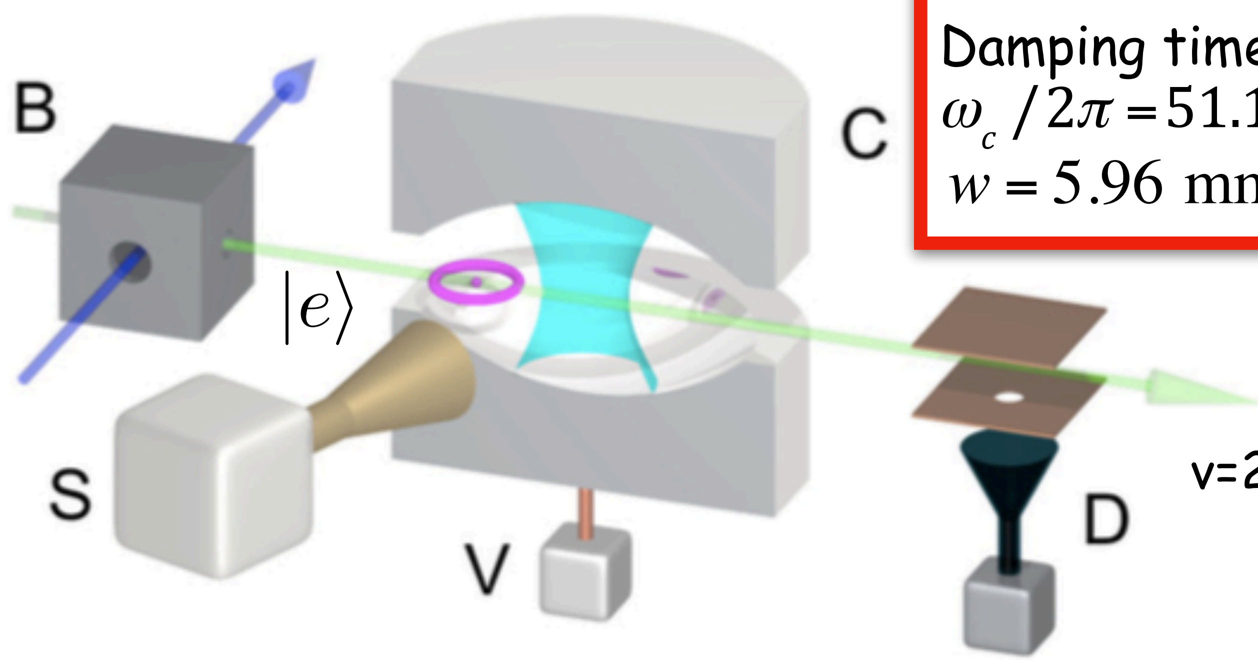
Switch on resonant interaction

Switch off resonant interaction

Field β to be measured is injected into the cavity at t=0

$$|\alpha\rangle = e^{-\alpha^2/2} \sum_n (\alpha^n / \sqrt{n!}) \, |n\rangle$$

Experimental procedure



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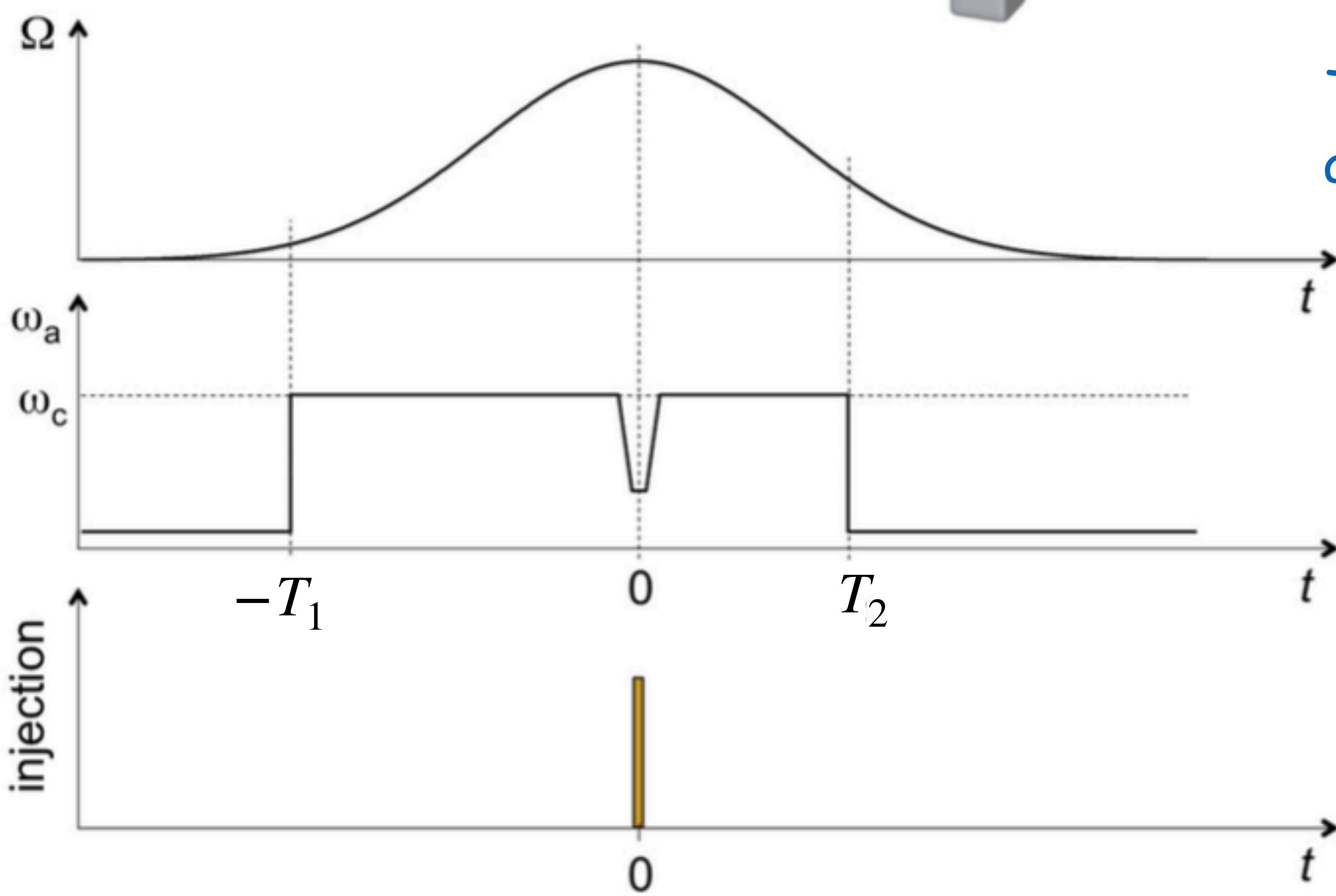


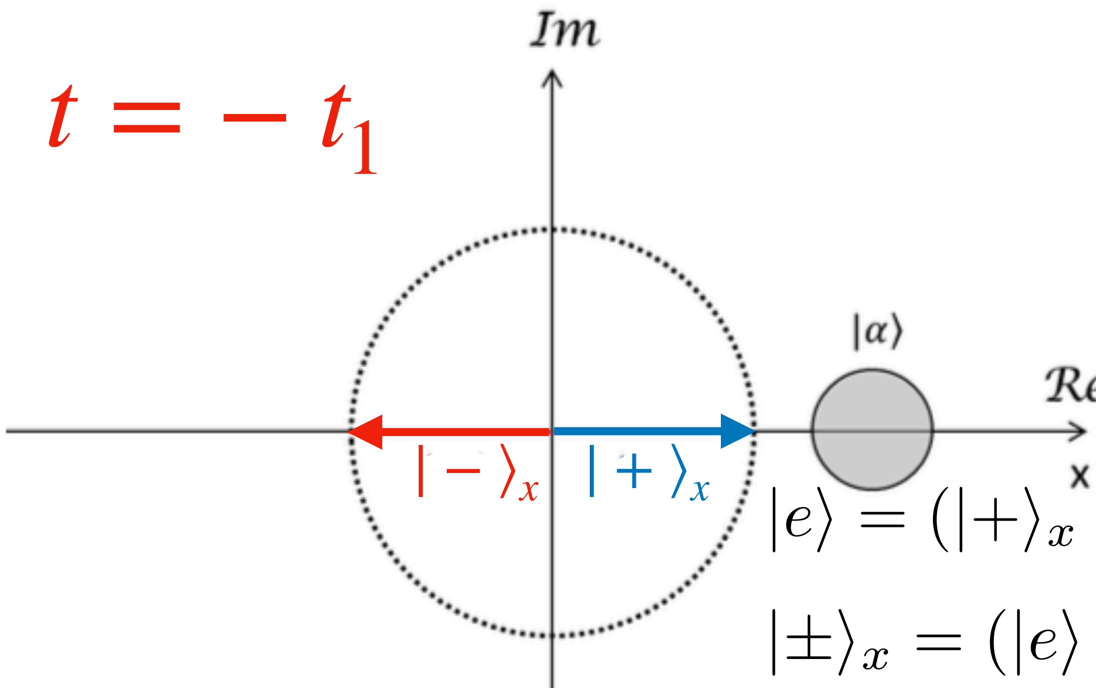
Temporal variation of the atom-cavity coupling

Modulation of atomic frequency —> induces π phase shift between $|e\rangle$and$|g\rangle$
—> time inversion!

Field β to be measured is injected into the cavity at t=0

Measurement protocol

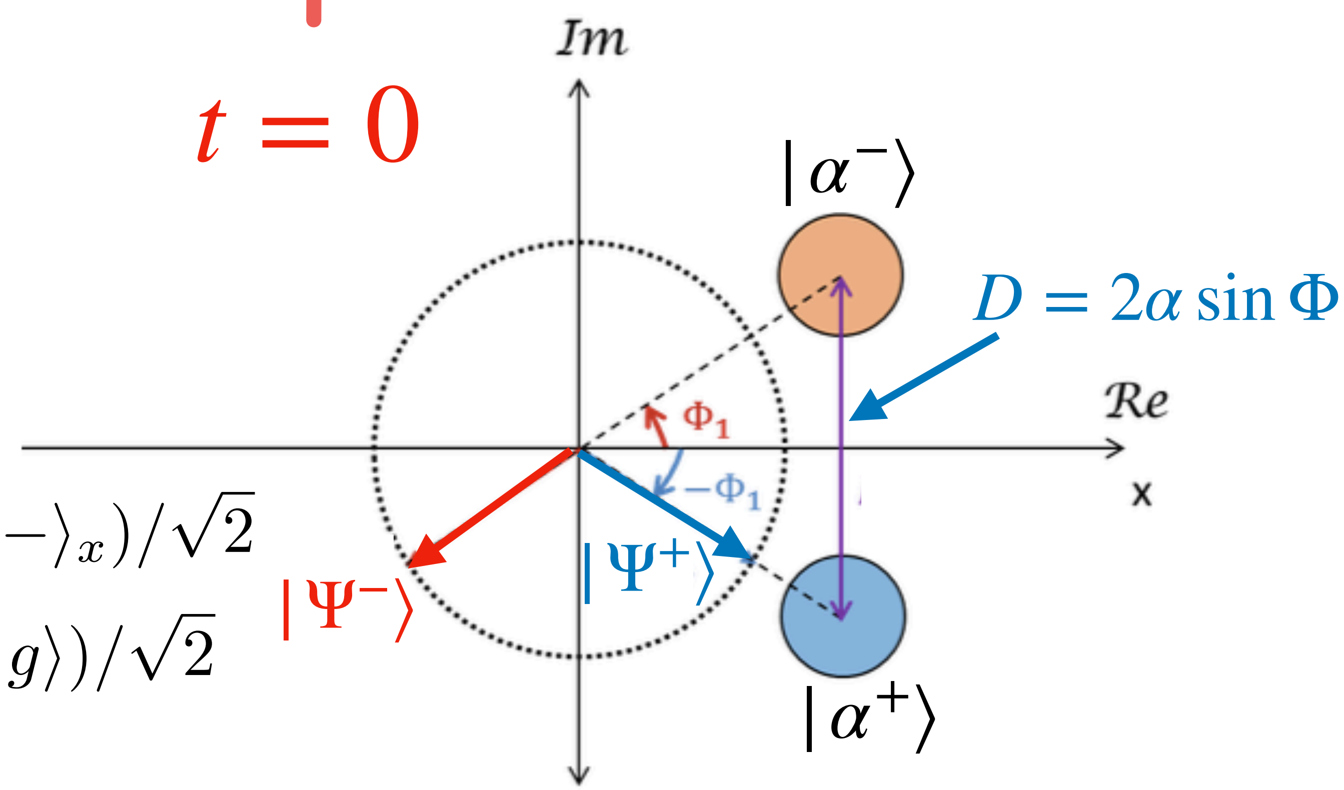
$t = -t_1$

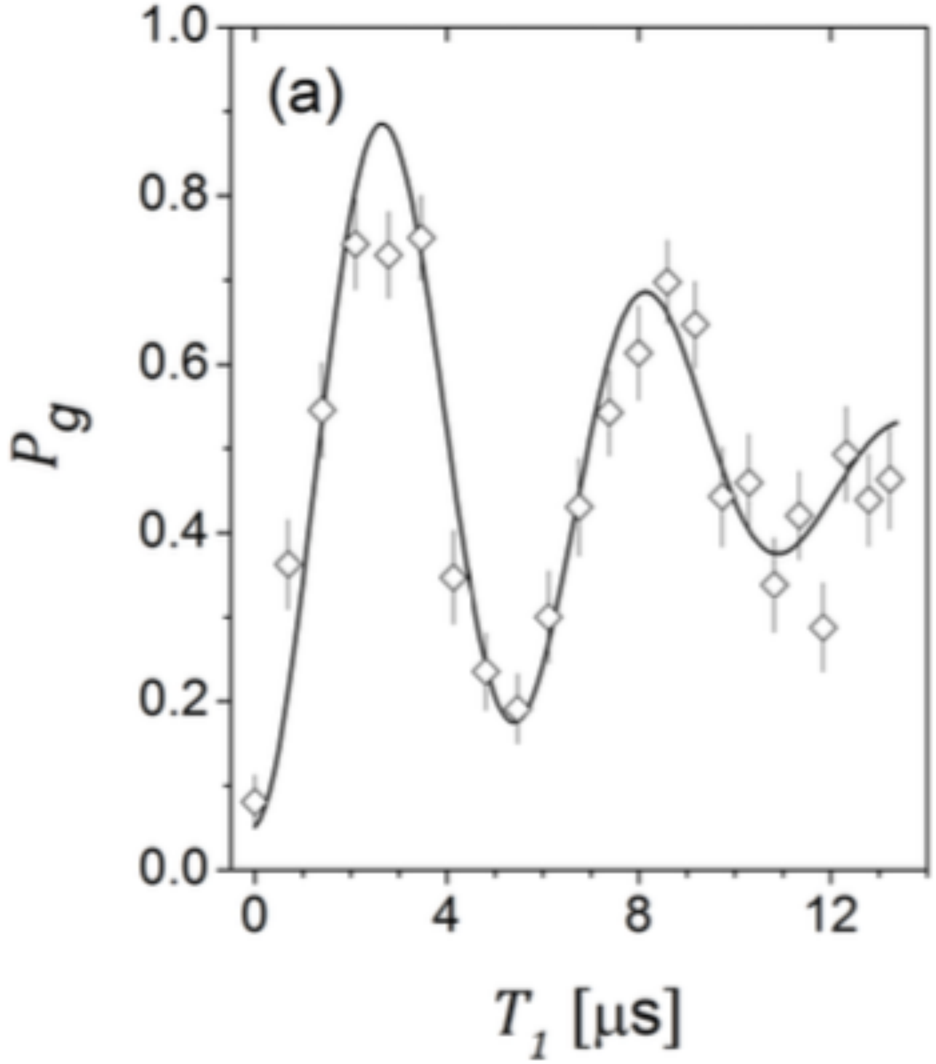


$$|e\rangle = (|+\rangle_x + |-\rangle_x)/\sqrt{2}$$

$$|\pm\rangle_x = (|e\rangle \pm |g\rangle)/\sqrt{2}$$

$t = 0$





$$|\Psi\rangle \approx \frac{1}{\sqrt{2}}\left[e^{-i\Phi_1\alpha^2}|\alpha^+\rangle|\Psi^+\rangle - e^{i\Phi_1\alpha^2}|\alpha^-\rangle|\Psi^-\rangle\right]$$

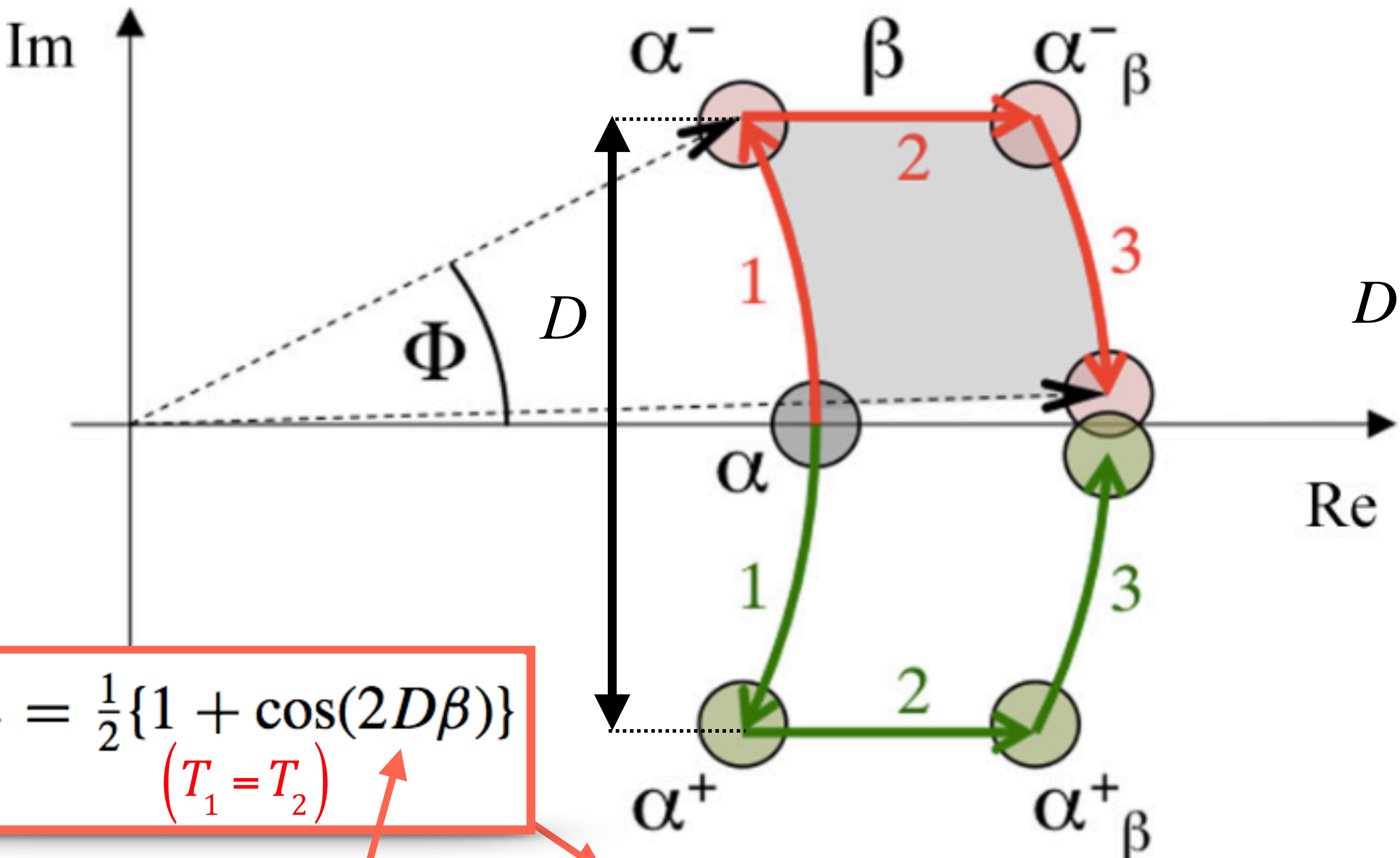
$$|\alpha^\pm\rangle = |\alpha e^{\mp i\Phi_1}\rangle$$

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}\left[e^{\mp i\Phi_1}|e\rangle \pm |g\rangle\right]$$

$$\Phi_1 = \Omega_0 T_1/4\alpha$$

(α large)

Measurement protocol



$$D = 2\alpha \sin\Phi$$

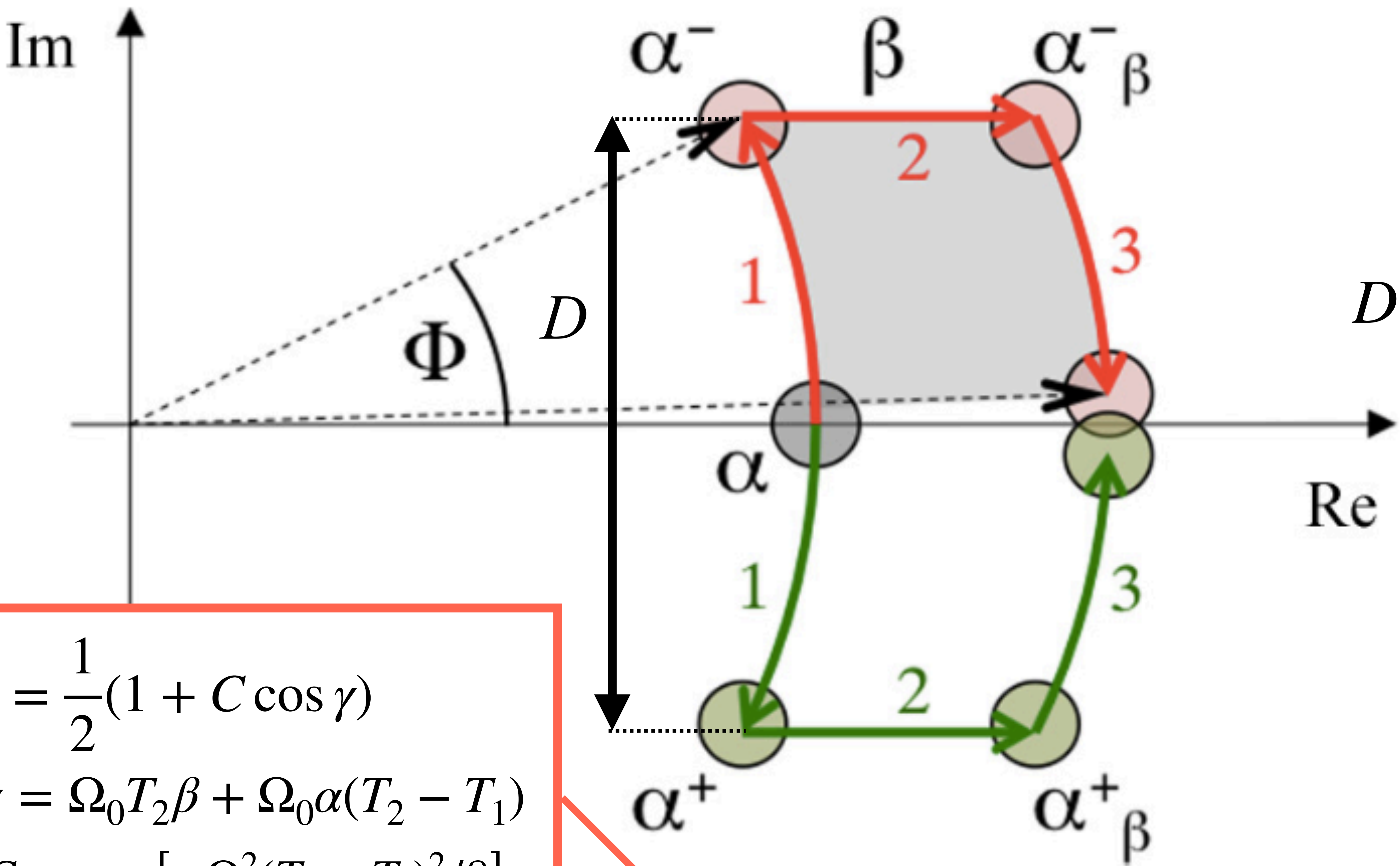
$$P_g = \tfrac{1}{2}\{1 + \cos(2D\beta)\}$$

$$(T_1 = T_2)$$

Geometric phase

$$F(\beta) \equiv \sum_{g,e} P_j(\beta) \left(\frac{d \ln\left[P_j(\beta)\right]}{d\beta} \right)^2 \Rightarrow \Delta\beta \geq 1/\sqrt{\nu F(\beta)}$$

Measurement protocol



$$D = 2\alpha \sin\Phi$$

$$P_g = \frac{1}{2}(1 + C\cos\gamma)$$

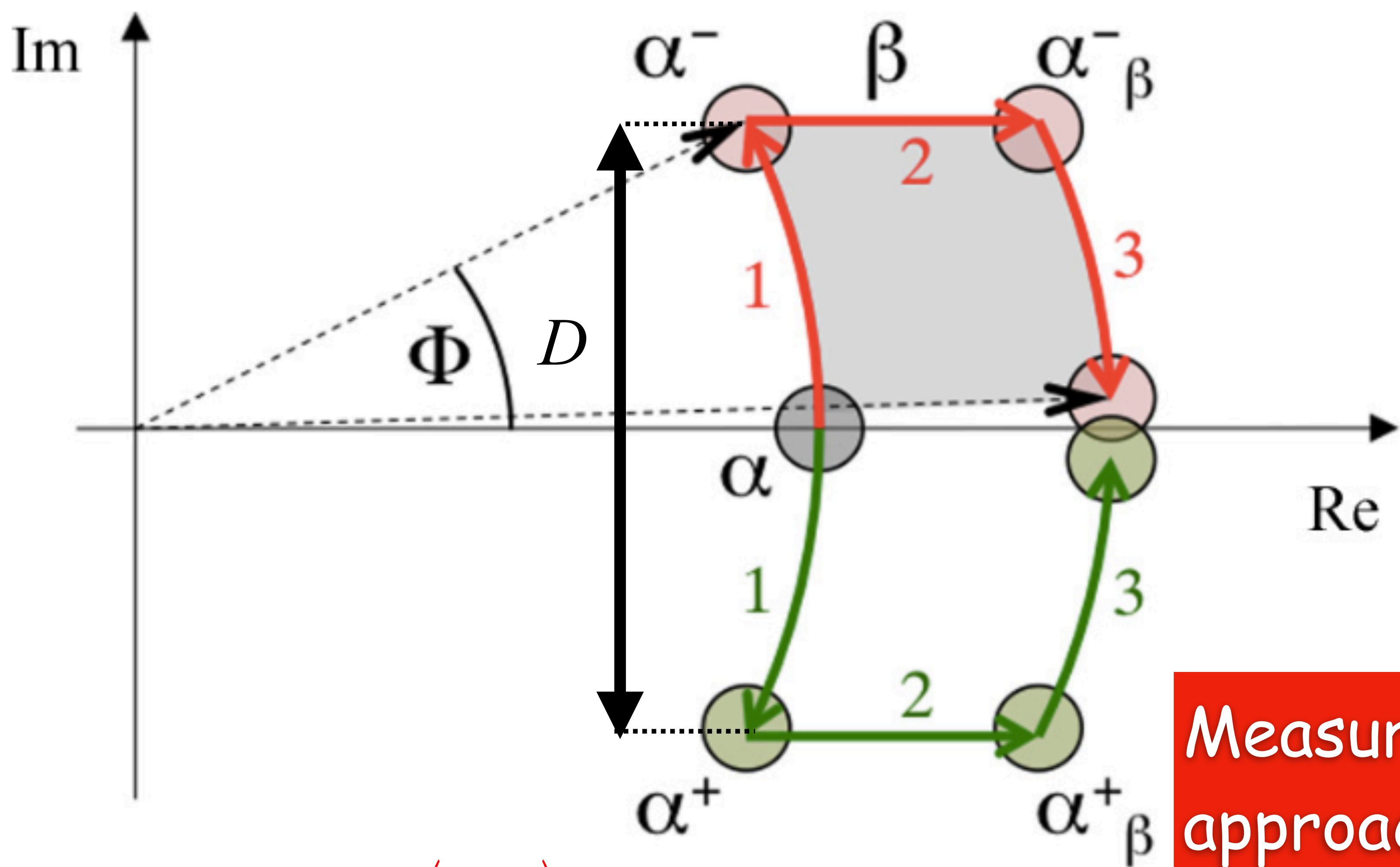
$$\gamma = \Omega_0 T_2 \beta + \Omega_0 \alpha (T_2 - T_1)$$

$$C = \exp\left[-\Omega_0^2 (T_1 - T_2)^2/8\right]$$

$$F(\beta) \equiv \sum_{g,e} P_j(\beta) \left(\frac{d \ln\left[P_j(\beta)\right]}{d\beta} \right)^2 \Rightarrow \Delta\beta \geq 1/\sqrt{\nu F(\beta)}$$

Better to have large T_2
but $\Omega_0^2 (T_2 - T_1)^2/8 \ll 1$

Measurement protocol



$$D = 2\alpha \sin\Phi$$

$$\hat{D}(\beta) = e^{\beta(\hat{a}^\dagger - \hat{a})} \Rightarrow \hat{h} = -i(\hat{a}^\dagger - \hat{a})$$

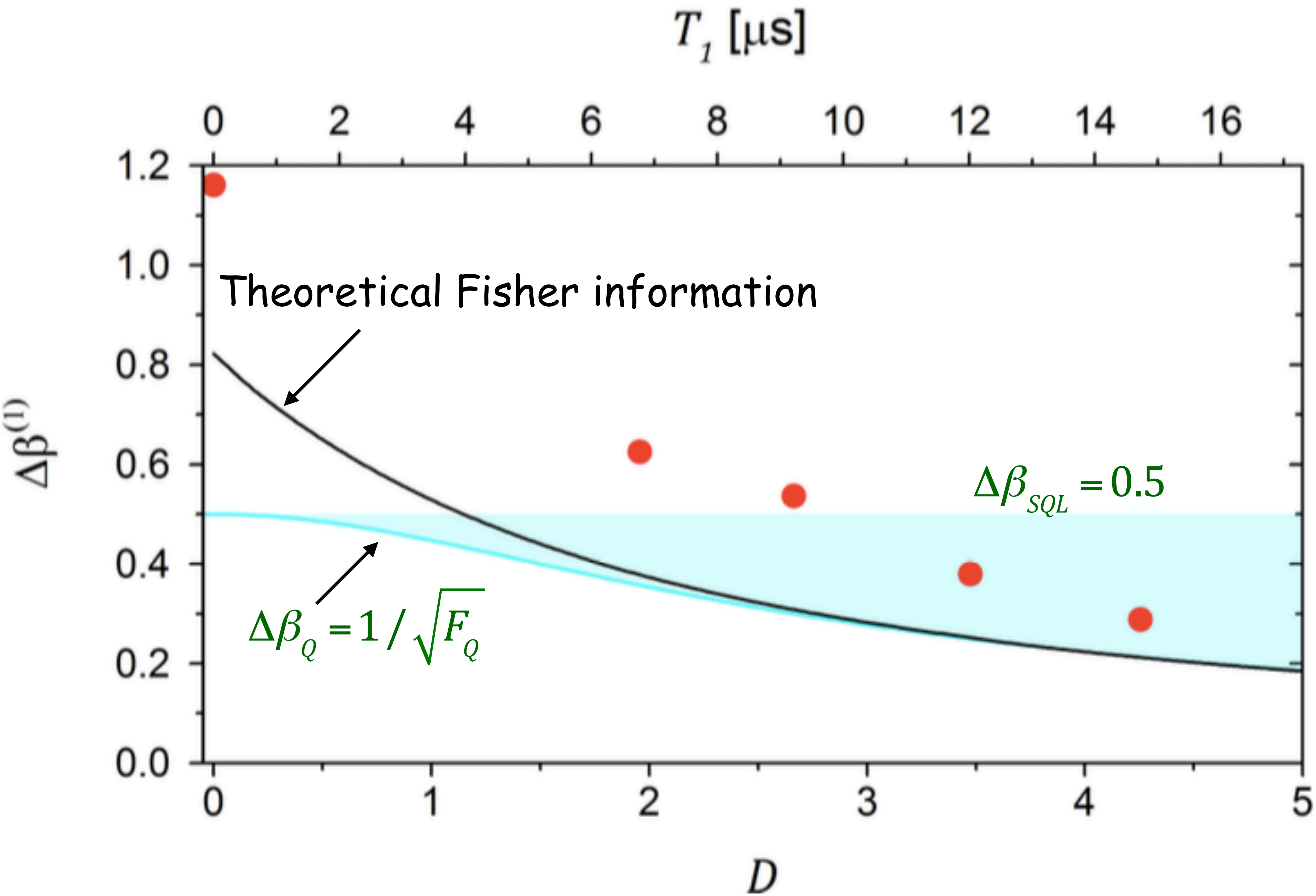
$$\mathcal{F}_Q = 4\langle(\Delta\hat{h})^2\rangle = 4(1 + D^2)$$

Measured Fisher information approaches the quantum Fisher information limit for large enough values of D (the difference is below 1.8% for D>2)

Coherent state: D=0 —> $\mathcal{F}_Q = 4$ —> Standard quantum limit: $\Delta\beta_{SQL} = 1/\sqrt{F(\beta)} = 0.5$

Maximum value: D=2 —> $\mathcal{F}_Q = 4(1 + 4\alpha^2) \approx 6\alpha^2 \Rightarrow$ Heisenberg scaling

Experimental results



Best result: $F_{exp} = 3SQL$

$$10\log_{10}\left(\sqrt{F_{\exp}/F_{SQL}}\right) \approx 2.4\ \text{dB}$$

QUANTUM METROLOGY IN LOSSY SYSTEMS

RECALLING: QUANTUM FISHER INFORMATION

In the first lecture, we defined, for a given measurement corresponding to the POVM $\{\hat{E}(\xi)\}$, the Fisher information,

$$F[X;\{\hat{E}(\xi)\}] = \int d\xi\, p(\xi|X)\left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2 = \int d\xi \frac{1}{p(\xi|X)}\left[\frac{\partial p(\xi|X)}{\partial X}\right]^2$$

and we have also defined the "Quantum Fisher information," which is obtained by maximizing the above expression with respect to all quantum measurements:

$$\mathcal{F}_Q(X) = \max_{\{\hat{E}(\xi)\}} F[X;\{\hat{E}(\xi)\}]$$

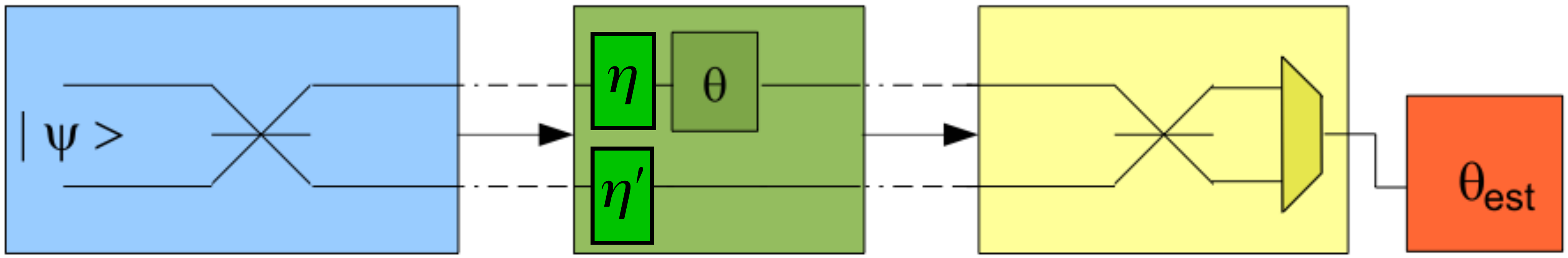
The lower bound for the precision in the measurement of the parameter X is then $\sqrt{\langle(\Delta X_{\rm est})^2\rangle} \geq 1/\sqrt{N\mathcal{F}_Q(X)}$, where N is the number of repetitions of the experiment.

The quantum Fisher information for pure states that evolve according to $|\psi(X)\rangle = \hat{U}(X)|\psi(0)\rangle$, where X is the parameter to be estimated and $\hat{U}(X)$ is a unitary operator, is

$$\boxed{\mathcal{F}_Q(X) = 4\langle(\Delta\hat{H})^2\rangle_0\,, \quad \langle(\Delta\hat{H})^2\rangle_0 \equiv \langle\psi(0)|\left[\hat{H}(X) - \langle\hat{H}(X)\rangle_0\right]^2|\psi(0)\rangle}$$

where $\hat{H}(X) \equiv i\frac{d\hat{U}^\dagger(X)}{dX}\hat{U}(X) = -i\hat{U}^\dagger(X)\frac{d\hat{U}(X)}{dX}$

Parameter estimation with losses



Loss of a single photon transforms NOON state into a separable state!

$$|\psi(N)\rangle = \frac{|N,0\rangle + |0,N\rangle}{\sqrt{2}} \rightarrow |N-1,0\rangle \ \text{ or } \ |0,N-1\rangle$$

No simple analytical expression for Fisher information!

For small N, more robust states can be numerically calculated

Experimental test with more robust states (for N=2):

nature photonics

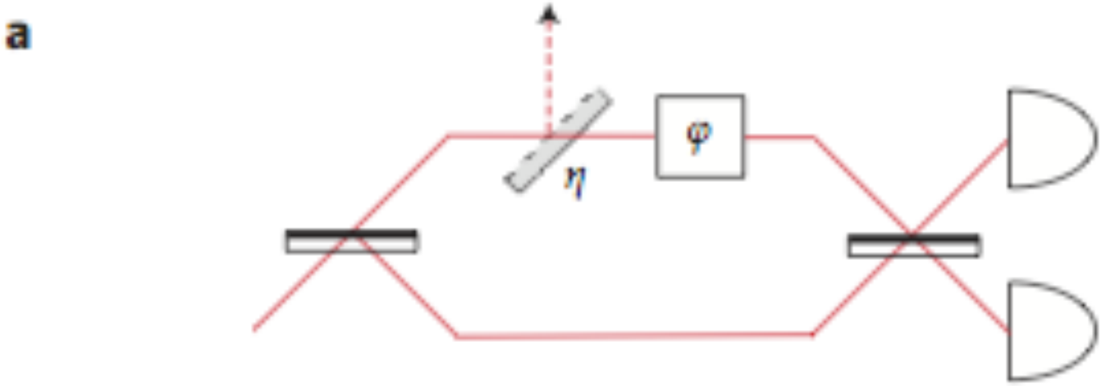
LETTERS

PUBLISHED ONLINE: 4 APRIL 2010 | DOI: 10.1038/NPHOTON.2010.39

Experimental quantum-enhanced estimation of a lossy phase shift

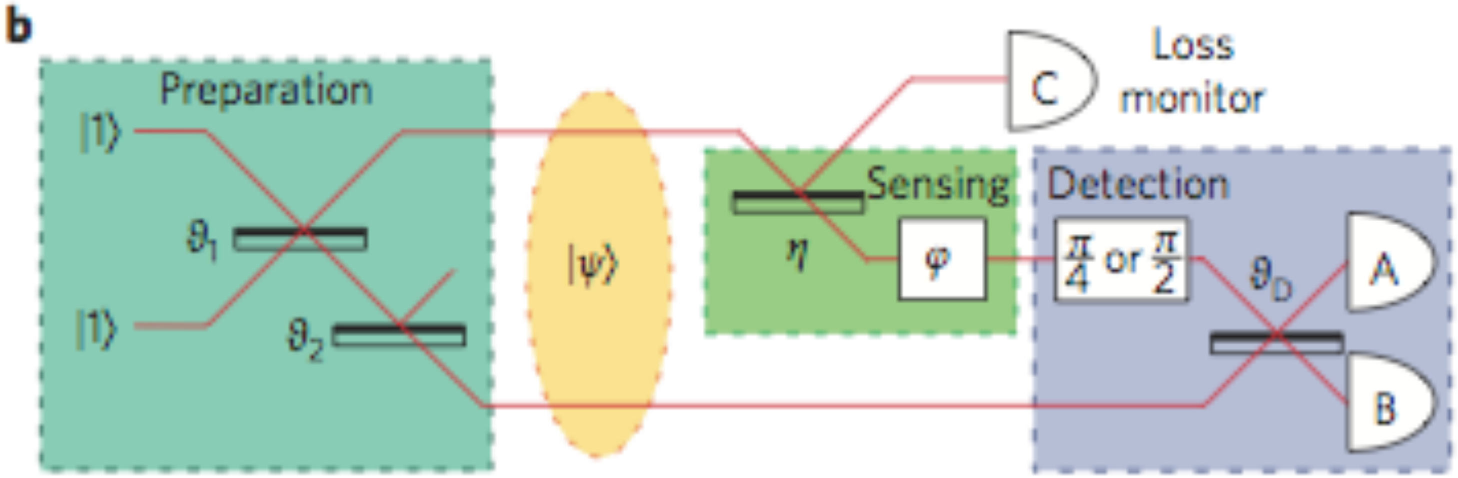
M. Kacprowicz[1], R. Demkowicz-Dobrzański[1,2]*, W. Wasilewski[2], K. Banaszek[1,2] and I. A. Walmsley[3]

Parameter estimation with losses - experiments

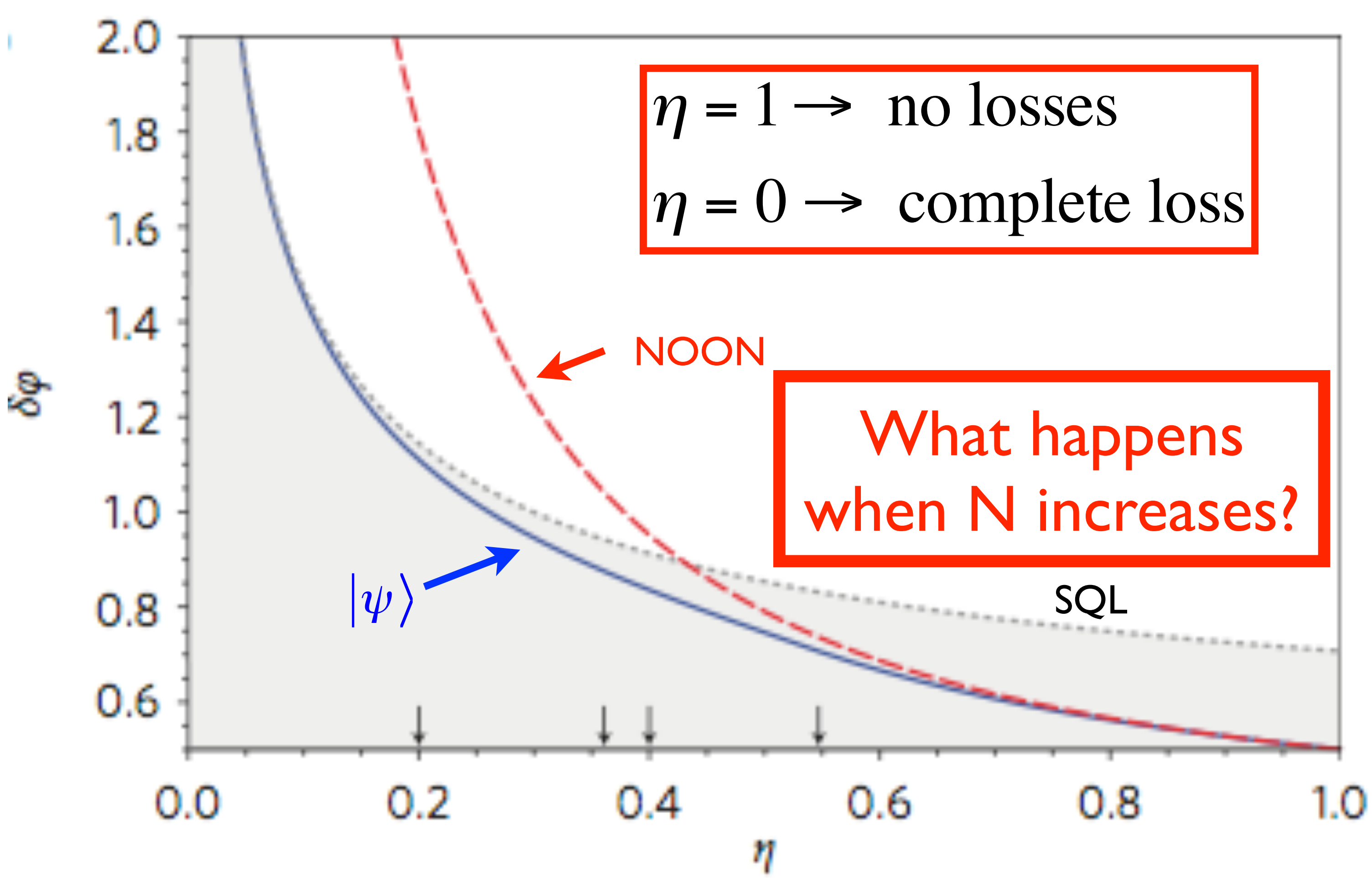


States leading to minimum uncertainty in the presence of noise:

$$|\psi\rangle = \sqrt{x_2}\,|20\rangle + \sqrt{x_1}\,|11\rangle - \sqrt{x_0}\,|02\rangle$$



Coefficients are determined numerically for each value of η. Losses simulated by a beam splitter in the upper arm. These states are prepared by two beam splitters.



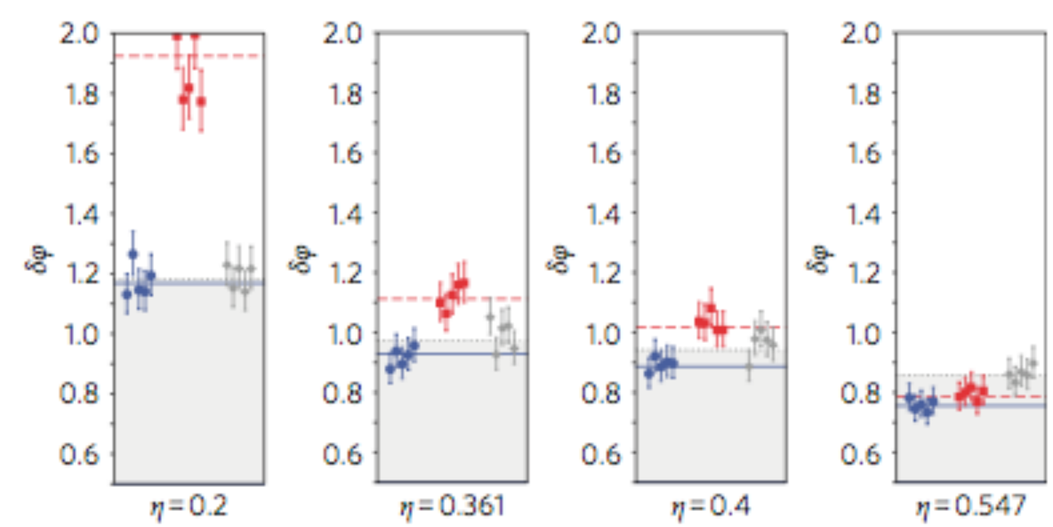


Figure 5 | Uncertainty of phase estimates. Uncertainties obtained using two-photon optimal (circles) and NOON (squares) states, as well as attenuated laser pulses in the SIL regime (diamonds), rescaled by the square root of the number of coincidences. For each transmission η, data are shown for five phases $\varphi = 0, \pm 0.2, \pm 0.4$ rad. Horizontal lines represent the theoretical Cramér-Rao bounds for given classes of input states, taking into account imperfections of the interferometer.

Parameter estimation with losses - theory

C. W. Helstrom, Quantum detection and estimation theory (Academic Press, New York, 1976); A. S. Holevo, Probabilistic and statistical aspects of quantum theory (North-Holland, Amsterdam, 1982); S. L. Braunstein and C. M. Caves, PRL 72, 3439 (1994).

We have now (Asymptotically attainable when $N \to \infty$)

$$\delta X \geq 1/\sqrt{N\mathscr{F}_Q\left[\hat{\rho}\left(X_{\text{real}}\right)\right]}, \quad \mathscr{F}_Q\left(\hat{\rho}\right) \equiv \max_{\hat{E}_j} F\left(\hat{\rho},\hat{E}_j\right)$$

$$F\left(\hat{\rho},\hat{E}_j\right) \equiv \sum_j p_j(X)\left(\frac{d\,\ln\left[p_j(X)\right]}{dx}\right)^2, \quad p_j(X) = \mathrm{Tr}\left[\hat{\rho}(X)\hat{E}_j\right]$$

$$\hat{L}_{ij} = \frac{2}{p_i + p_j}[\partial\hat{\rho}(X)/\partial X]_{ij},$$

p_i, p_j eigenvalues of $\hat{\rho}$

General expression for the quantum Fisher information:

$\mathcal{F}_Q[\hat{\rho}(X)] = \mathrm{Tr}\left[\hat{\rho}(X)\hat{L}^2(X)\right]$ where the operator $\hat{L}$ ("symmetric logarithmic derivative") is defined by the equation $\dfrac{d\hat{\rho}(X)}{dX} = \dfrac{\hat{\rho}(X)\hat{L}(X) + \hat{L}(X)\hat{\rho}(X)}{2}$

For pure states:

$$\hat{\rho}^2 = \hat{\rho}\,, \frac{d\hat{\rho}(X)}{dX} = \frac{d\hat{\rho}^2(X)}{dX} = \hat{\rho}(X)\frac{d\hat{\rho}(X)}{dX} + \frac{d\hat{\rho}(X)}{dX}\hat{\rho}(X) \Rightarrow \hat{L}(X) = 2\frac{d\hat{\rho}(X)}{dX}$$

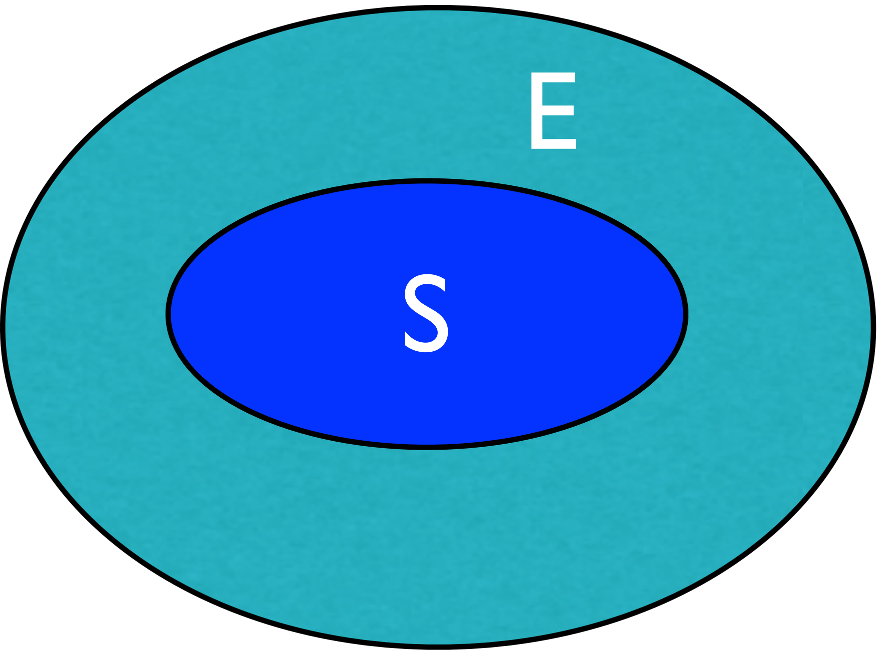
so that, from $\hat{\rho}(X) = \hat{U}(X)\hat{\rho}(0)\hat{U}^\dagger(X)$, one gets the previous result $\mathcal{F}_Q(X) = 4\langle(\Delta\hat{H})^2\rangle_0$, with $\hat{H}(X) \equiv i\frac{d\hat{U}^\dagger(X)}{dX}\hat{U}(X)$.

General case: $\hat{L}$ difficult to evaluate - analytic expression not known.

Parameter estimation in open systems: Extended space approach

B. M. Escher, R. L. Matos Filho, and L. D., **Nature Physics** 7, 406 (2011); **Braz. J. Phys.** 41, 229 (2011)

Given initial state and non-unitary evolution, define in S+E



$$|\Phi_{S,E}(x)\rangle = \hat{U}_{S,E}(x)|\psi\rangle_S|0\rangle_E$$ (Purification)

Then

$$\mathscr{F}_Q \equiv \max_{\hat{E}_j^{(S)}\otimes\hat{1}} F\left(\hat{E}_j^{(S)}\otimes\hat{1}\right) \leq \max_{\hat{E}_j^{(S,E)}} F\left(\hat{E}_j^{(S,E)}\right) \equiv \mathscr{C}_Q$$

since measurements on S+E should yield more information than measurements on S alone.

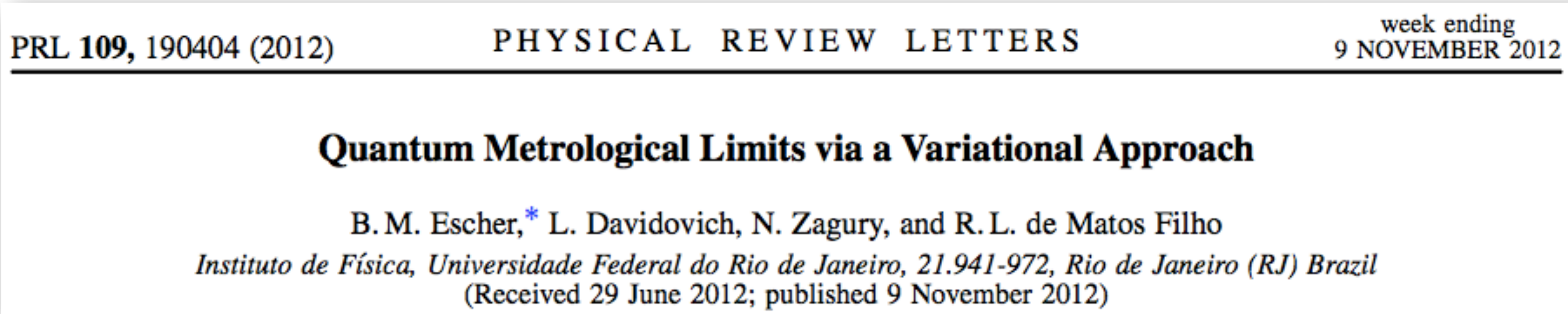
Physical meaning of this bound: information obtained about parameter when S+E is monitored

Least upper bound: Minimization over all unitary evolutions in S+E - difficult problem

Bound is attainable - there is always a purification such that $\mathscr{C}_Q = \mathscr{F}_Q$

Then, monitoring S+E yields same information as monitoring S

Minimization procedure


PRL 109, 190404 (2012) PHYSICAL REVIEW LETTERS week ending 9 NOVEMBER 2012

Quantum Metrological Limits via a Variational Approach

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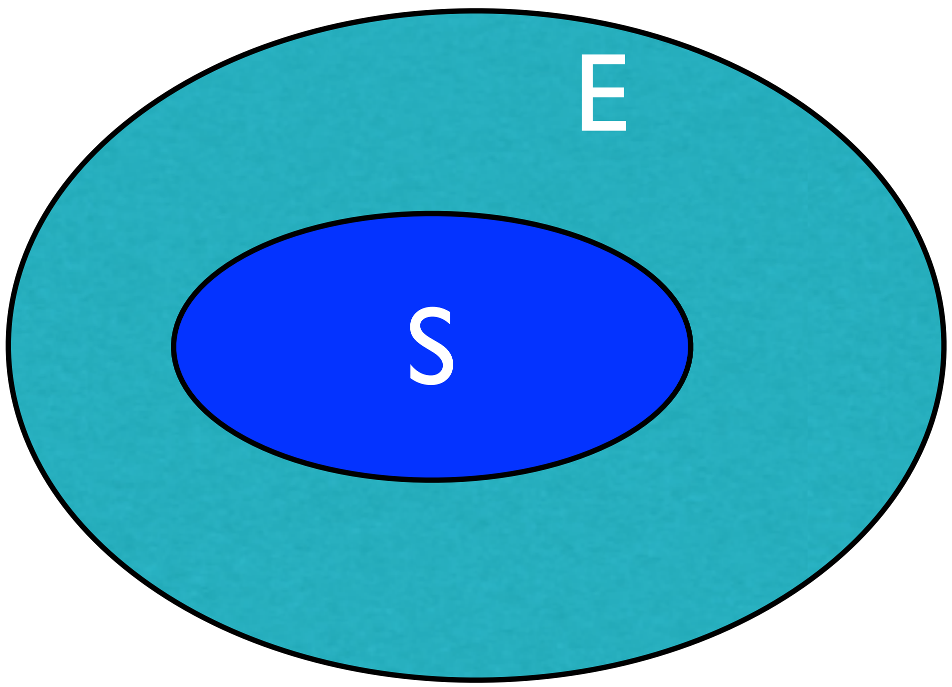
(Received 29 June 2012; published 9 November 2012)



There is always an unitary operator acting only on E that connects two different purifications of ρ_S

Given $|\Phi_{S,E}(x)\rangle = \hat{U}_{S,E}(x)|\psi\rangle_S|0\rangle_E$,

$$i\frac{d|\Phi_{S,E}(x)\rangle}{dx} = \hat{H}_{S,E}(x)|\Phi_{S,E}(x)\rangle,$$

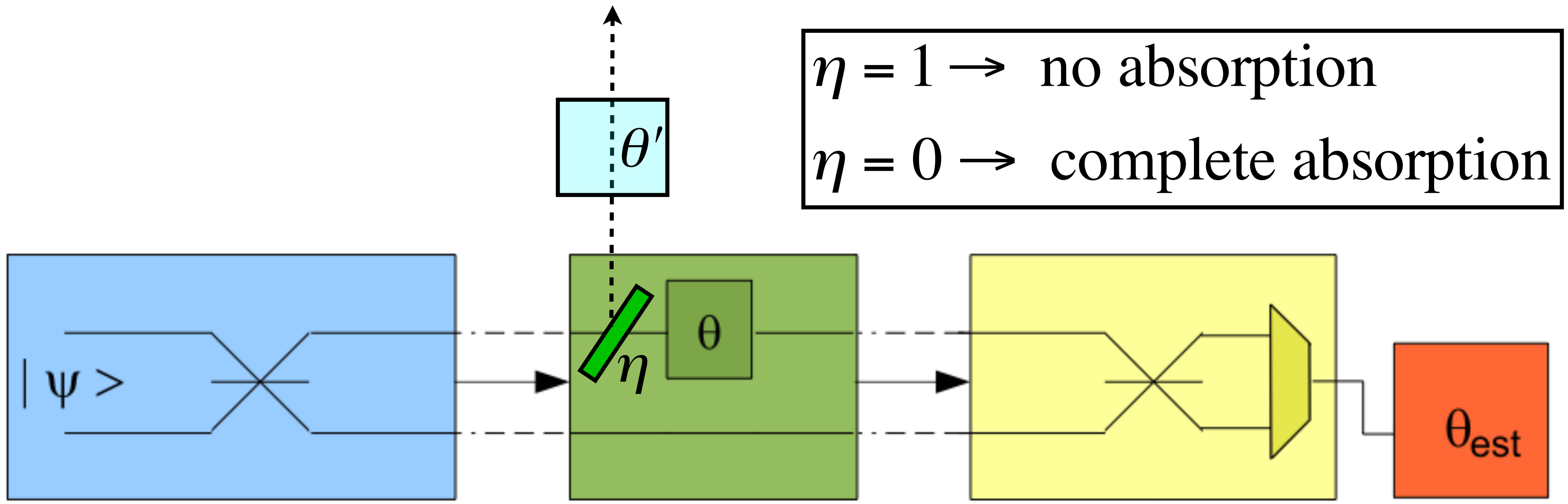
then any other purification can be written as:

$$|\Psi_{S,E}(x)\rangle = u_E(x)|\Phi_{S,E}(x)\rangle$$

Define $\hat{h}_E(x) = i\frac{d\hat{u}_E^\dagger(x)}{dx}\hat{u}_E(x)$

Minimize now C_Q over all Hermitian operators $h_E(x)$ that act on E. Above paper proposes iterative procedure for doing this.

Quantum limits for lossy optical interferometry



One uses here a similar strategy: a phase displacement on the environment so as to remove additional information on the phase θ.

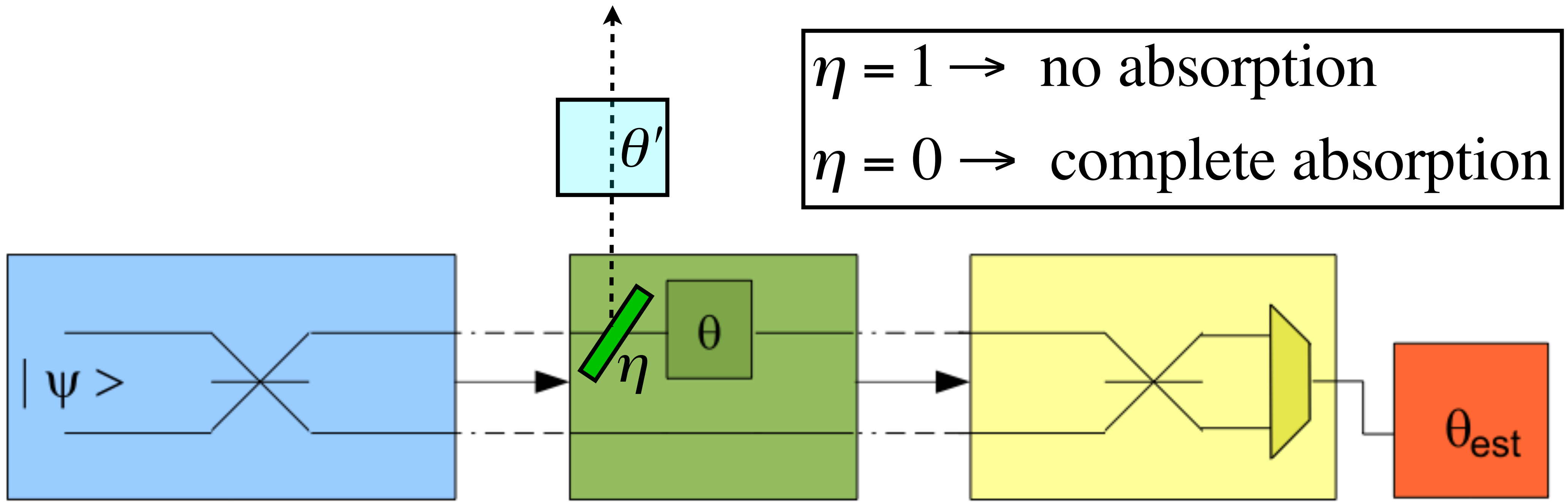
Minimization of the quantum Fisher information of system + environment yields an upper bound for the Fisher information of the system:

$$\mathcal{C}_Q(\hat{\rho}_0) = \frac{4\eta\langle\hat{n}\rangle_0 \Delta^2\hat{n}_0}{(1-\eta)\Delta^2\hat{n}_0 + \eta\langle\hat{n}\rangle_0}$$

Note that if $(1-\eta)\Delta^2\hat{n}_0 \ll \eta\langle\hat{n}\rangle_0$ then $\mathcal{C}_Q \rightarrow \Delta^2\hat{n}_0$, the quantum Fisher information for pure states. On the other hand, in the high-dissipation limit $\eta \ll 1$, one has $(1-\eta)\Delta^2\hat{n}_0 \gg \eta\langle\hat{n}\rangle_0$, yielding a standard-limit scaling:

$$\delta\theta \geq \sqrt{(1-\eta)/4\eta\langle\hat{n}\rangle_0}$$

Quantum limits for lossy optical interferometry



States with well-defined total photon number: $|\psi_0\rangle = \sum_{n=0}^{N} \beta_n |n, N-n\rangle$

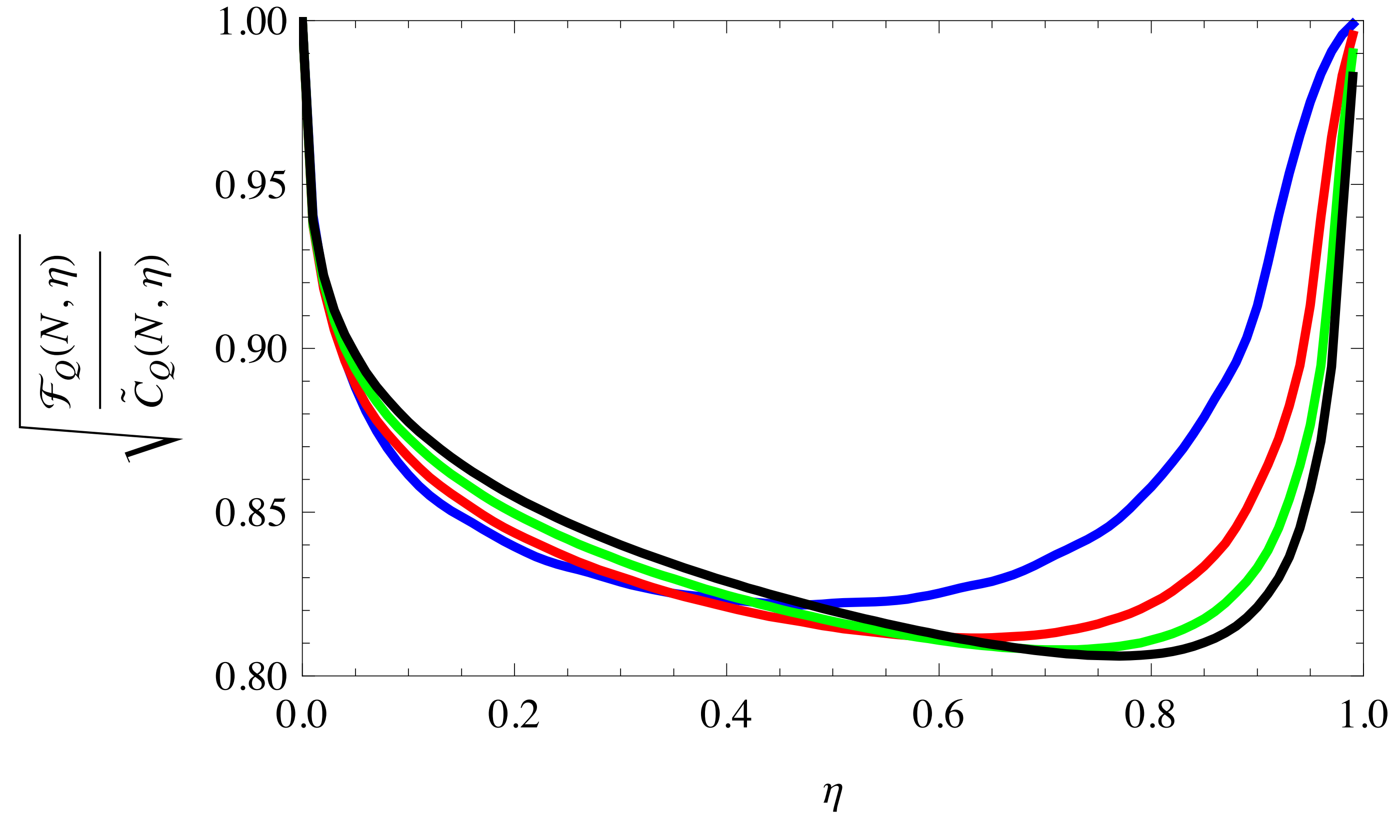
$$2\delta\theta \geq \left[1 + \sqrt{1 + \frac{1-\eta}{\eta} N}\right] / N$$

$N \ll \frac{\eta}{1-\eta} \Rightarrow \sqrt{\nu}\delta\theta \geq 1/N$ —>Heisenberg scaling

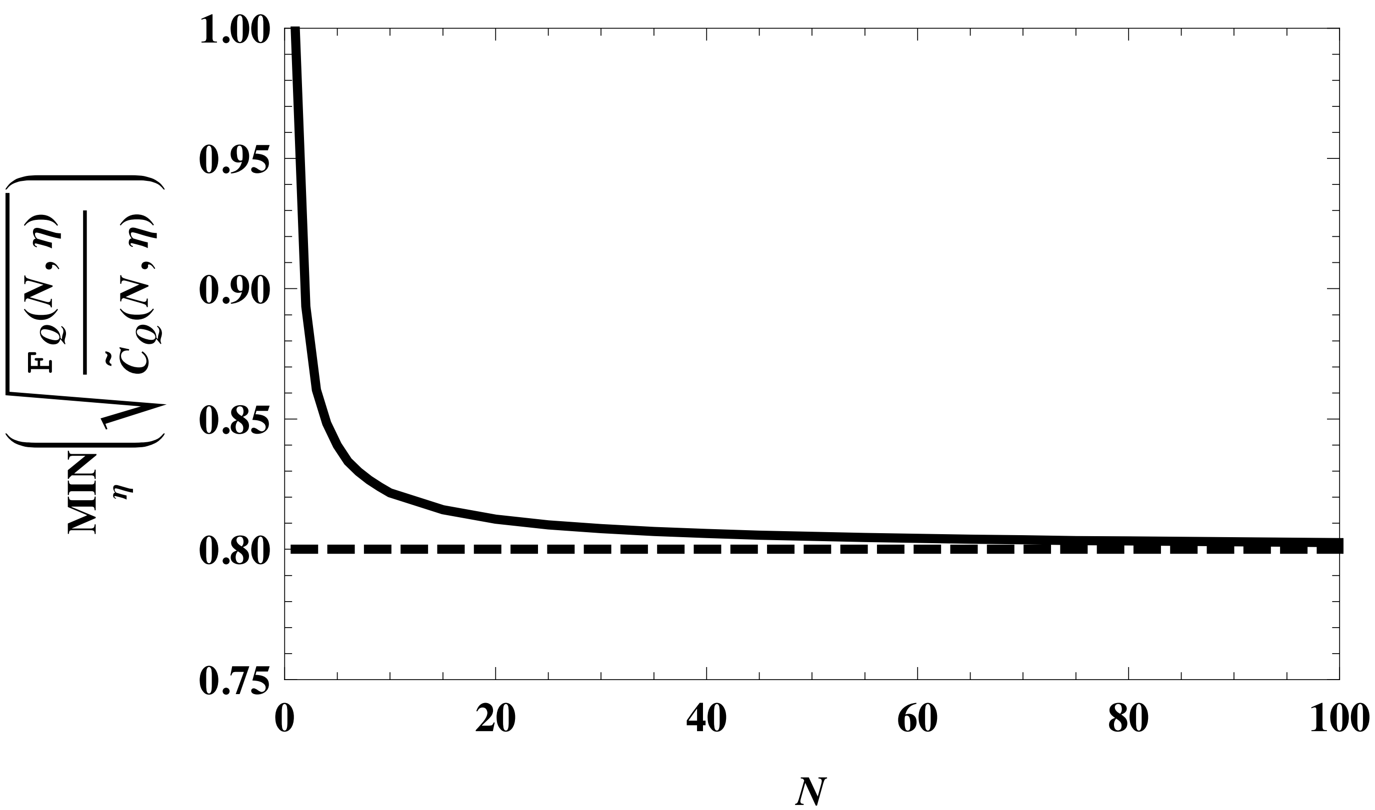
$N \gg \frac{\eta}{1-\eta} \Rightarrow \delta\theta \geq \frac{\sqrt{1-\eta}}{2\sqrt{\nu\eta N}}$ —>Standard scaling

For N sufficiently large, $1/\sqrt{N}$ behavior is always reached!

How good is this bound?



Comparison between the numerical maximum value of $\mathscr{F}_Q$ and the upper bound $\mathscr{C}_Q$ as a function of η, for $N = 10$ (blue), $N = 20$ (red), $N = 30$ (green), and $N = 40$ (black).



Behavior of the minimum for all values of η, as a function of N

Phase diffusion in optical interferometer

PRL **109**, 190404 (2012) PHYSICAL REVIEW LETTERS week ending 9 NOVEMBER 2012

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(Received 29 June 2012; published 9 November 2012)

$$\dot{\rho} = \Gamma \mathcal{L}[a^\dagger a]\rho, \quad \mathcal{L}[O]\rho = 2O\rho O^\dagger - O^\dagger O\rho - \rho O^\dagger O$$

$$\Rightarrow \rho(t) = \sum_{m.n} e^{-\beta^2(n-m)^2}\rho_{n,m}(0)|n\rangle\langle m|, \quad \beta = \Gamma t$$

Possible purification:

Radiation pressure

Ground state of mirror (harmonic oscillator)

$$|\Phi_{S,E}(\phi)\rangle = e^{-i\phi\hat{n}_S} e^{i(2\beta)\hat{n}_S\hat{x}_\mathrm{E}}|\psi_S\rangle|0_E\rangle \Rightarrow C_Q = 4\Delta n^2$$ Trivial!

Choose instead:

$$|\Phi_{S,E}(\phi)\rangle = e^{i\phi\lambda\hat{p}_\mathrm{E}/(2\beta)} e^{-i\phi\hat{n}_S} e^{i(2\beta)\hat{n}_S\hat{x}_\mathrm{E}}|\psi_S\rangle|0_E\rangle$$

$$\Rightarrow C_Q = (1-\lambda)^2 4\Delta n^2 + \lambda^2/(2\beta^2)$$

$\lambda \longrightarrow$ Variational parameter

Phase diffusion in optical interferometer

$$\delta\phi_{pd} \geq \sqrt{\frac{1}{\nu}\left(\frac{1}{4\Delta n^2} + 2\beta^2\right)}$$

Intrinsic quantum feature

Phase diffusion

Very close to numerical value obtained by Genoni, Olivares, and Paris for Gaussian state - PRL 106, 153603 (2011)

For Gaussian states:

$$\Delta n^2 \leq 2N(N+1)$$

(N is the average photon number)

Then:

$$C_Q^{\mathrm{opt}} \leq C_Q^{\max} \equiv \left[2\beta^2 + \frac{1}{8N(N+1)}\right]^{-1}$$

Comparison with numerical results

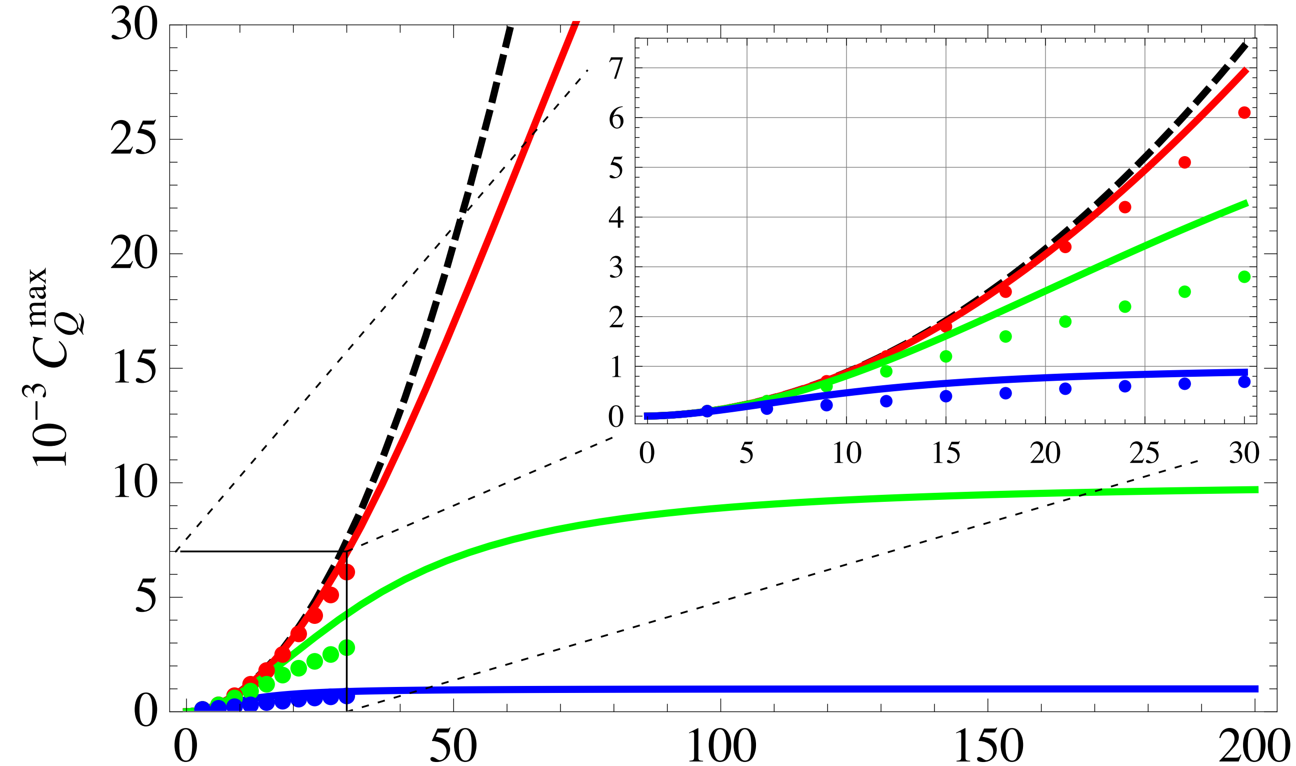
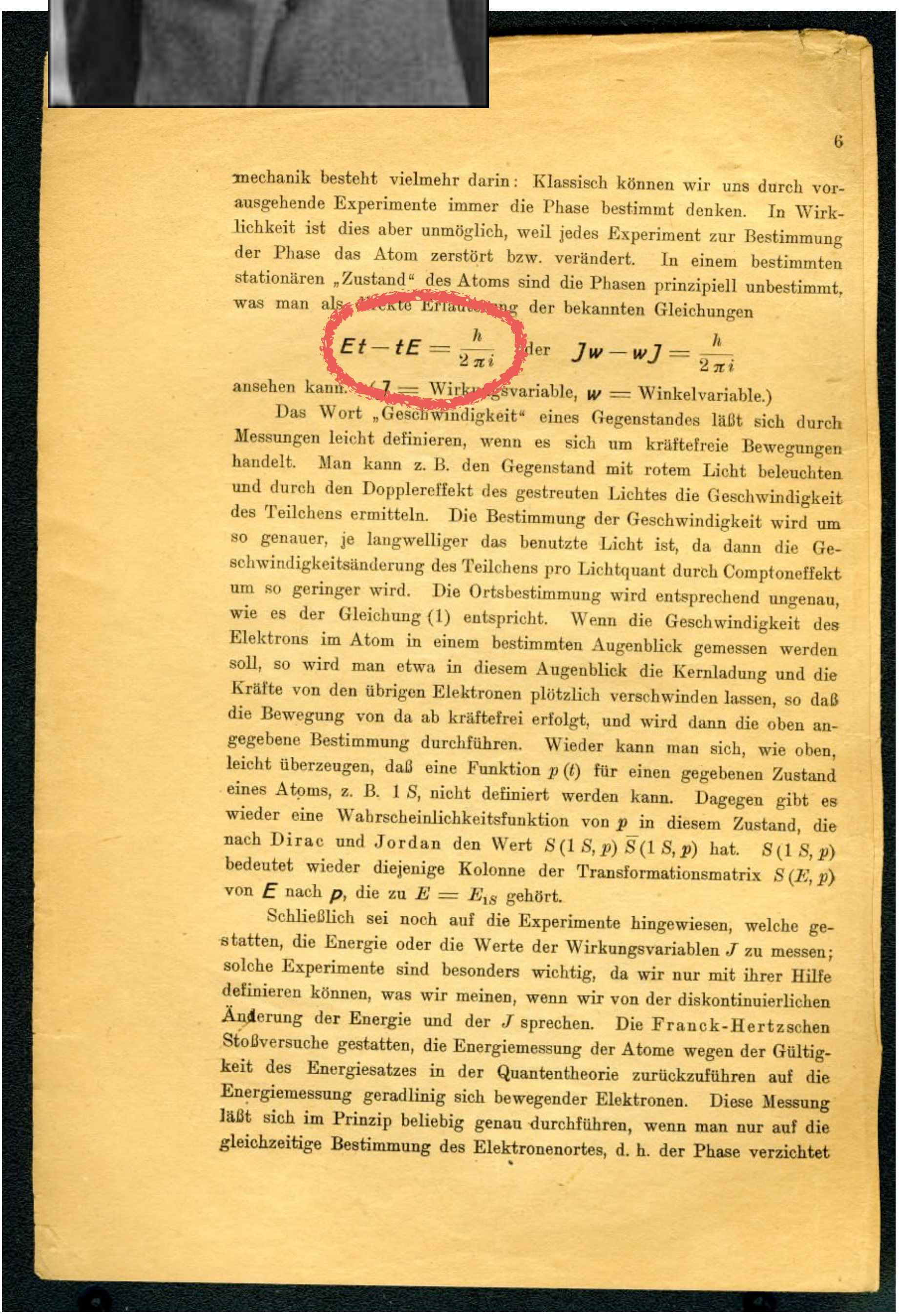


FIG. 1 (color online). Comparison between upper bound $C_Q^{\max}$ and the maximum quantum Fisher information $\mathcal{F}_Q^{\max}$ in Ref. [14] as a function of the average number of photons N. The dots stand for the values obtained in Ref. [14], the dashed line corresponds to the noiseless case ($\beta^2 = 0$), and the full lines correspond to $C_Q^{\max}$. The inset displays the two quantities up to $N = 30$, which was the range considered in Ref. [14]. From bottom to top, $\beta^2 = 5 \times 10^{-4}; 5 \times 10^{-5}; 5 \times 10^{-6}$.

Energy-time uncertainty

$$\Delta E \Delta T \geq \hbar$$



6

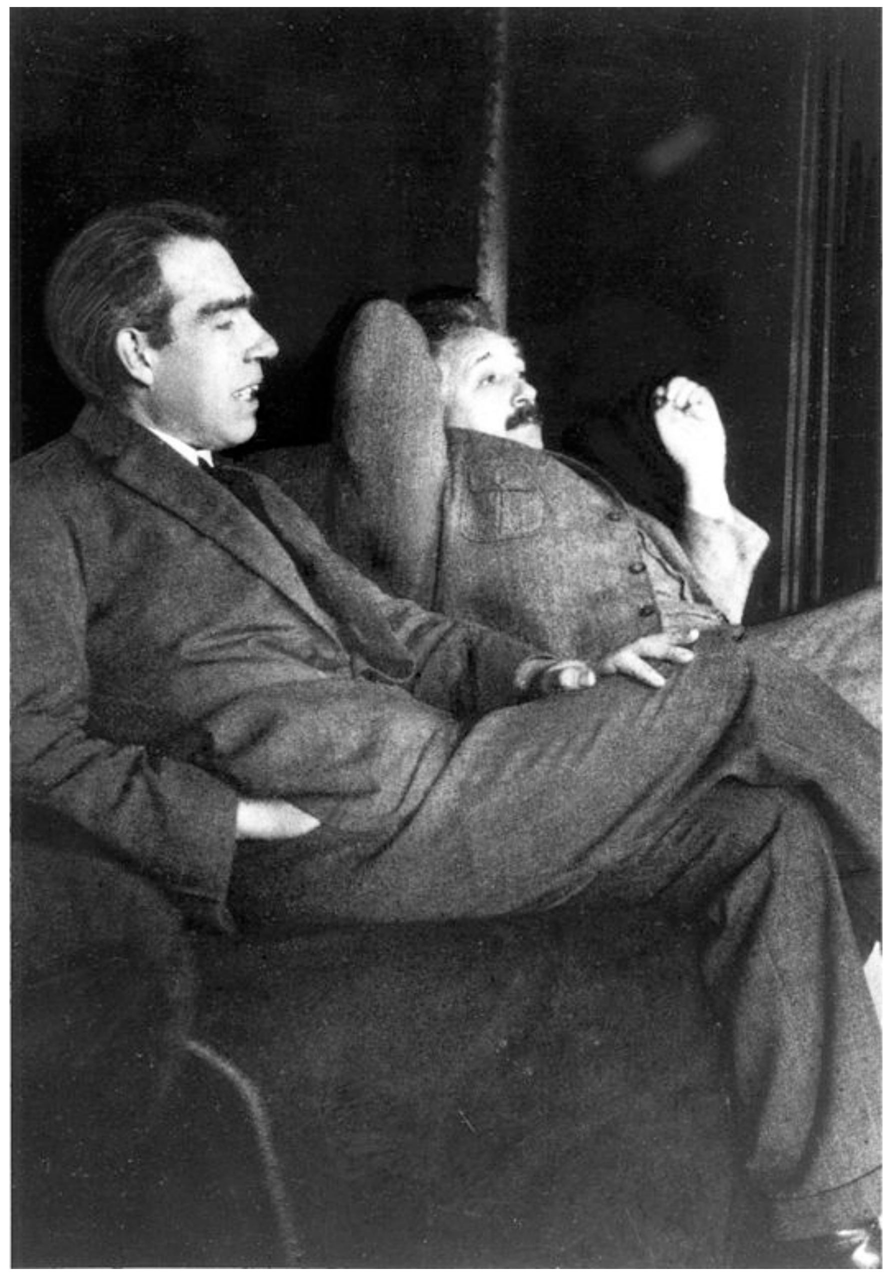
mechanik besteht vielmehr darin: Klassisch können wir uns durch vorausgehende Experimente immer die Phase bestimmt denken. In Wirklichkeit ist dies aber unmöglich, weil jedes Experiment zur Bestimmung der Phase das Atom zerstört bzw. verändert. In einem bestimmten stationären „Zustand" des Atoms sind die Phasen prinzipiell unbestimmt, was man als direkte Erläuterung der bekannten Gleichungen

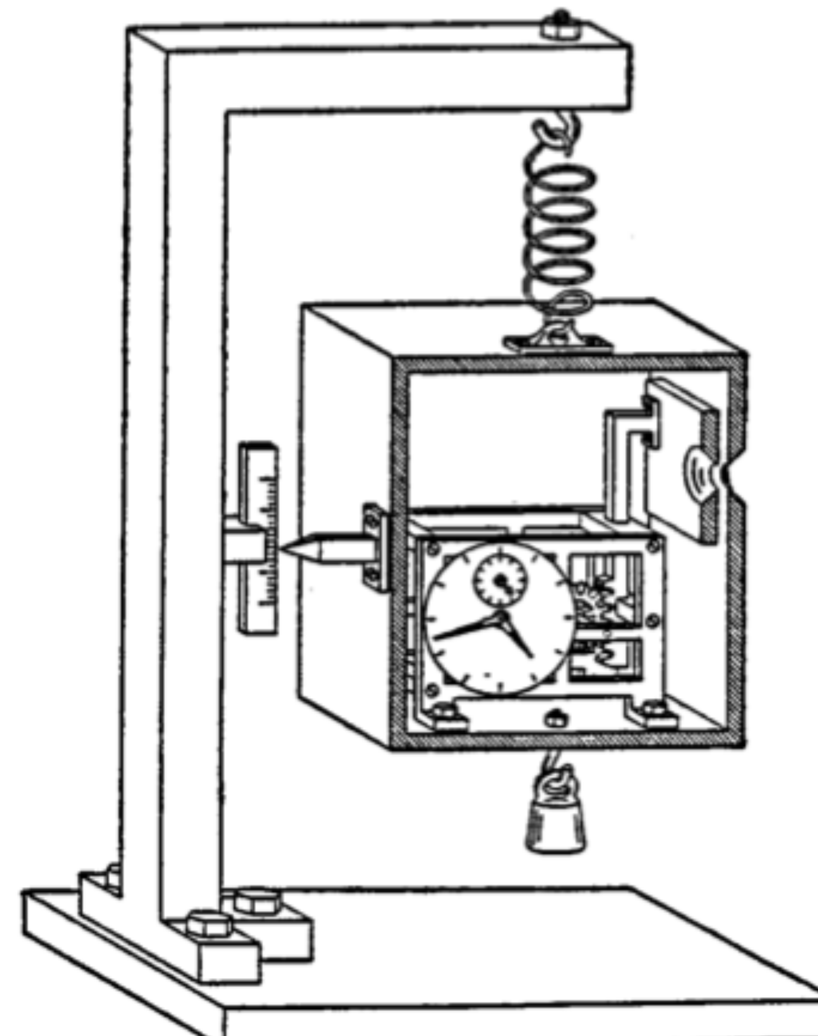
$$Et - tE = \frac{h}{2\pi i} \quad \text{oder} \quad Jw - wJ = \frac{h}{2\pi i}$$

ansehen kann. (J = Wirkungsvariable, w = Winkelvariable.)

Das Wort „Geschwindigkeit" eines Gegenstandes läßt sich durch Messungen leicht definieren, wenn es sich um kräftefreie Bewegungen handelt. Man kann z. B. den Gegenstand mit rotem Licht beleuchten und durch den Dopplereffekt des gestreuten Lichtes die Geschwindigkeit des Teilchens ermitteln. Die Bestimmung der Geschwindigkeit wird um so genauer, je langwelliger das benutzte Licht ist, da dann die Geschwindigkeitsänderung des Teilchens pro Lichtquant durch Comptoneffekt um so geringer wird. Die Ortsbestimmung wird entsprechend ungenau, wie es der Gleichung (1) entspricht. Wenn die Geschwindigkeit des Elektrons im Atom in einem bestimmten Augenblick gemessen werden soll, so wird man etwa in diesem Augenblick die Kernladung und die Kräfte von den übrigen Elektronen plötzlich verschwinden lassen, so daß die Bewegung von da ab kräftefrei erfolgt, und wird dann die oben angegebene Bestimmung durchführen. Wieder kann man sich, wie oben, leicht überzeugen, daß eine Funktion $p(t)$ für einen gegebenen Zustand eines Atoms, z. B. $1\,S$, nicht definiert werden kann. Dagegen gibt es wieder eine Wahrscheinlichkeitsfunktion von p in diesem Zustand, die nach Dirac und Jordan den Wert $S(1\,S, p)\,\overline{S}(1\,S, p)$ hat. $S(1\,S, p)$ bedeutet wieder diejenige Kolonne der Transformationsmatrix $S(E, p)$ von E nach p, die zu $E = E_{1S}$ gehört.

Schließlich sei noch auf die Experimente hingewiesen, welche gestatten, die Energie oder die Werte der Wirkungsvariablen J zu messen; solche Experimente sind besonders wichtig, da wir nur mit ihrer Hilfe definieren können, was wir meinen, wenn wir von der diskontinuierlichen Änderung der Energie und der J sprechen. Die Franck-Hertzschen Stoßversuche gestatten, die Energiemessung der Atome wegen der Gültigkeit des Energiesatzes in der Quantentheorie zurückzuführen auf die Energiemessung geradlinig sich bewegender Elektronen. Diese Messung läßt sich im Prinzip beliebig genau durchführen, wenn man nur auf die gleichzeitige Bestimmung des Elektronenortes, d. h. der Phase verzichtet





Energy-time uncertainty

THE UNCERTAINTY RELATION BETWEEN ENERGY AND TIME IN NON-RELATIVISTIC QUANTUM MECHANICS

By L. MANDELSTAM * and Ig. TAMM

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(Received February 22, 1945)

A uncertainty relation between energy and time having a simple physical meaning is rigorously deduced from the principles of quantum mechanics. Some examples of its application are discussed.

1. Along with the uncertainty relation between coordinate q and momentum p one considers in quantum mechanics also the uncertainty relation between energy and time.

The former relation in the form of the inequality

$$\Delta q \cdot \Delta p \geqslant \frac{h}{2}, \qquad (1)$$

An entirely different situation is met with in the case of the relation

$$\Delta H \cdot \Delta T \sim h, \qquad (2)$$

where ΔH is the standard of energy, ΔT — a certain time interval, and the sign $\sim$ denotes that the left-hand side is at least of the order of the right-hand one.

Leonid Mandelstam

Igor Tamm

Energy-time uncertainty

Derivation of Mandelstam and Tamm is based on the relations:

$\Delta E \Delta A \geq \frac{1}{2} |\langle [H, A] \rangle|$, and $\hbar \frac{d\langle A \rangle}{dt} = i \langle [H, A] \rangle$, where A is an observable of the system ("clock observable"), not explicitly dependent on time, and H is the Hamiltonian that rules the evolution. From these two equations, we get:

$$\Delta E \Delta A \geq \frac{\hbar}{2} \left| \frac{d\langle A \rangle}{dt} \right| .$$

Integrating this equation with respect to time, and using that $\int_a^b |f(t)| dt \geq \left| \int_a^b f(t) dt \right|$, one gets

$$\boxed{\Delta E \Delta t \geq \frac{\hbar}{2} \left(\frac{|\langle A \rangle_{t+\Delta t} - \langle A \rangle_t|}{\overline{\Delta A}} \right)} ,$$

where $\overline{\Delta A} \equiv (1/\Delta t) \int_t^{t+\Delta t} \Delta A \, dt$ is the time average of ΔA over the integration region. We define the time interval ΔT as the shortest time for which the average value of A changes by an amount equal to its averaged standard deviation. Then $\boxed{\Delta E \Delta T \geq \hbar/2}$.

Energy-time uncertainty

Mandelstam and Tamm also presented a more accurate derivation, which is directly related to more modern treatments.

One starts again from

$$\Delta E \Delta A \geq \frac{\hbar}{2}\left|\frac{d\langle A\rangle}{dt}\right|.$$

Let us choose now A to be the projection operator onto the initial state: $A = P_0 = |\psi_0\rangle\langle\psi_0|$, so that $P_0^2 = P_0$ and

$\Delta P_0 = \sqrt{\langle P_0^2\rangle - \langle P_0\rangle^2} = \sqrt{\langle P_0\rangle - \langle P_0\rangle^2}$, which implies that

$$\Delta E \geq \frac{\hbar}{2}\left|\frac{d\langle P_0\rangle/dt}{\sqrt{\langle P_o\rangle - \langle P_0\rangle^2}}\right|.$$

Integrating this expression from 0 to τ, and using that $\int_a^b |f(t)|dt \geq \left|\int_a^b f(t)dt\right|$, one gets $\boxed{\Delta E \cdot \tau \geq \hbar \arccos\sqrt{\langle P_0\rangle_\tau}}$ where $\langle P_0\rangle_\tau = |\psi_0|\psi_\tau|^2$ is the fidelity between the initial and the final states.

Throughout this lecture, the image of arcos is defined in $[0,\pi]$. If the final state is orthogonal to the initial one, $\langle P_0\rangle_\tau = 0$ and $\boxed{\Delta E \cdot \tau \geq h/4.}$

Energy-time uncertainty

Note that the steps leading to $\Delta E \geq \frac{\hbar}{2}\left|\frac{d\langle P_0\rangle/dt}{\sqrt{\langle P_o\rangle - \langle P_0\rangle^2}}\right|$ also hold if H depends on time. Therefore, from this equation one may extract a more general expression:

$$\boxed{\int_0^\tau \Delta E(t)\, dt \geq \hbar \arccos\sqrt{F}}$$

which is an implicit bound for the time needed to reach a fidelity $F = |\langle\psi_0|\psi_\tau\rangle|^2$ between the initial and final state.

Energy-time uncertainty

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Geometry of Quantum Evolution

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Geometric derivation. Inequality derived from the condition that actual path followed by the states should be larger than geodesic connecting the two states.

Generalization to non-unitary processes? Life-time for decay processes? Hamiltonian should not show up!

Motivation

1. Foundations of quantum mechanics: How to interpret this relation? (Heisenberg, Einstein, Bohr, Mandelstam and Tamm, Landau and Peierls, Fock and Krylov, Aharonov and Bohm, Bhattacharyya)
2. Computation times: e.g., time taken to flip a spin — Quantum speed limit
3. Quantum-classical transition: Decoherence time
4. Control of the dynamics of a quantum system: find the fastest evolution given initial and final states and some restriction on the resources (e.g. the energy) or the general structure of the Hamiltonian.
5. Relation with quantum metrology

Quantum speed limit for physical processes

M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, PRL **110**, 050402 (2013)

The previous results imply an extension to open systems of the Mandelstam-Tamm relation:

$$\boxed{\arccos\sqrt{\Phi_B[\hat\rho(0),\hat\rho(\tau)]} \le \int_0^\tau \sqrt{\mathcal{F}_Q(t)}/2dt}$$

Bures length of geodesic

Bures length of actual path followed by state of the system

$\Rightarrow$ Lower bound for time needed to reach fidelity $\Phi_B[\hat\rho(0),\hat\rho(\tau)]$ between initial and final states

Special case: Unitary evolution, time-independent Hamiltonian, orthogonal states

Mandelstam-Tamm

$$\Phi_B\left[\hat\rho(0),\hat\rho(\tau)\right]=0,\quad \mathcal{F}_Q(t)=4\langle(\Delta H)^2\rangle/\hbar^2 \Rightarrow \boxed{\tau\sqrt{\langle(\Delta H)^2\rangle}\ge h/4}$$

Quantum speed limit for open systems: Purification procedure

$$\mathcal{D} := \arccos\sqrt{\Phi_B\left[\hat{\rho}(0),\hat{\rho}(\tau)\right]} \leq \int_0^\tau \sqrt{\mathcal{F}_Q(t)/4}\,dt$$

$\Downarrow$

Problem: No analytical expression for $\mathscr{F}_Q$

$\Rightarrow$ Purification!

$$\mathcal{D} \leq \int_0^\tau \sqrt{\mathcal{C}_Q(t)/4}\,dt = \int_0^\tau \sqrt{\langle\Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle}/\hbar\,dt.$$

$$\hat{\mathcal{H}}_{S,E}(t) := \frac{\hbar}{i}\frac{d\hat{U}^\dagger_{S,E}(t)}{dt}\hat{U}_{S,E}(t)$$

$\hat{U}_{S,E}(t)$: Evolution of purified state corresponding to $\hat{\rho}_S$

Quantum speed limit for physical processes: amplitude damping channel

As seen in Lecture 2, the amplitude-damping channel may be described by the following equations (states without indices refer to the system — e.g. a two-level atom with $|1\rangle$ and $|0\rangle$ being the excited and ground states):

$$|0\rangle|0\rangle_E \to |0\rangle|0\rangle_E \,,$$

$$|1\rangle|0\rangle_E \to \sqrt{P(t)}|1\rangle|0\rangle_E + \sqrt{1-P(t)}|0\rangle|1\rangle_E \qquad P(t) = \exp(-\gamma t)$$

This is a quite natural, physically motivated purification of the evolution of two-level atom. The unitary evolution corresponding to this map is

$$\hat{U}_{S,E}(t) = \exp[-i\Theta(t)(\hat{\sigma}_+\hat{\sigma}_-^{(E)} + \hat{\sigma}_-\hat{\sigma}_+^{(E)})] \quad \hat{\sigma}_+|0\rangle = |1\rangle \,, \quad \hat{\sigma}_-|1\rangle = |0\rangle \,, \quad \hat{\sigma}_\pm^2 = 0$$

$$\hat{\sigma}_+\hat{\sigma}_- = |1\rangle\langle 1|$$

with $\Theta(t) = \arccos\sqrt{P(t)}$.

From this and $\mathcal{D} \leq \int_0^\tau \sqrt{\mathcal{C}_Q(t)/4}\, dt = \int_0^\tau \sqrt{\langle \Delta\hat{\mathcal{H}}_{S,E}^2(t)\rangle}/\hbar\, dt.$

one gets: $\mathcal{D} \leq \sqrt{\langle \hat{\sigma}_+\hat{\sigma}_-\rangle}\arccos[\exp(-\gamma t/2)]$

Initial population of excited state

Quantum speed limit for physical processes: amplitude damping channel (2)

This implies a lower bound for the distance-dependent decay time:

$$\mathcal{D} \le \sqrt{\langle \hat{\sigma}_+ \hat{\sigma}_- \rangle}\,\arccos[\exp(-\gamma\tau/2)] \Rightarrow \boxed{\gamma\tau \ge 2\ln\sec(\mathcal{D}/\sqrt{\langle \hat{\sigma}_+ \hat{\sigma}_- \rangle})}$$

Initial population of excited state

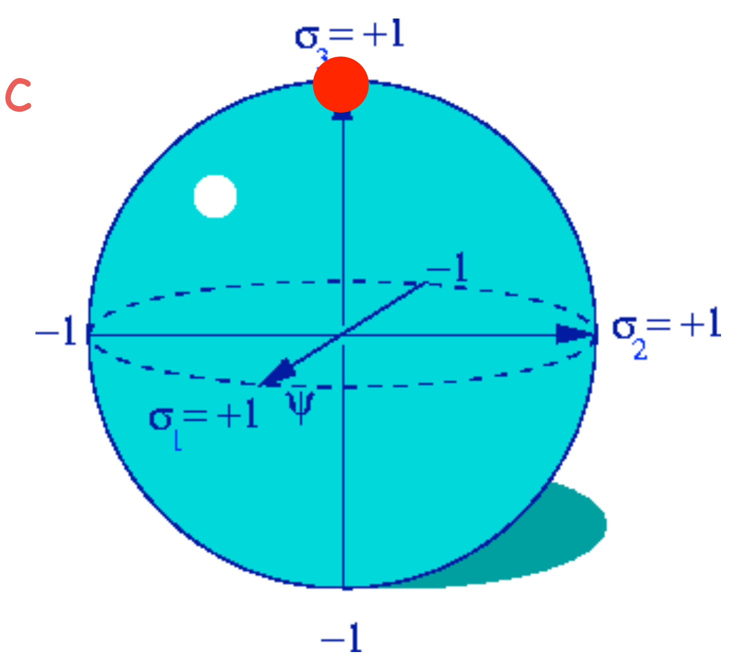
Bound is saturated if $\langle \hat{\sigma}_+ \hat{\sigma}_- \rangle = 0 \;\text{ or }\; 1$

$$\langle \hat{\sigma}_+ \hat{\sigma}_- \rangle = 1 \Rightarrow \; |1\rangle\langle 1| \to P(t)|1\rangle\langle 1| + [1 - P(t)]|0\rangle\langle 0|$$

$$\Rightarrow \Phi = \sqrt{P(\tau)} \Rightarrow \mathcal{D} = \arccos[\exp(-\gamma\tau/2)]$$

Interpretation:

If initial state is the excited state, then evolution is along a geodesic



Time for getting at the origin:

$$\Phi = 1/2, \;\; \mathcal{D} = \arccos(\Phi) = \pi/3, \;\; \gamma\tau = 2\ln 2 \approx 1.39$$

Time for getting deexcited:

$$\mathcal{D} = \pi/2 \Rightarrow \tau = \infty!$$