General estimation theory

We have shown that it is possible to win over the shot noise in optical interferometry, by using states with specific quantum features, like states with well-defined number of photons or squeezed states. In these examples, the estimation was obtained through measurement of the difference of photon numbers in the outgoing arms of the interferometer. It is not clear whether these are the best possible measurements, or whether better bounds can be obtained by using other incoming states.

One may ask whether it is possible to find general bounds and strategies for reaching them, which could be applied to many different systems, and could eventually help us to identify which are the best states and the best measurements for achieving the best possible precision.

This is the aim of this series of lectures: to develop, and apply to examples, a general estimation theory, capable not only to consider unitary evolutions of closed systems, like the one described here for the optical interferometer, but also open (noisy) systems.

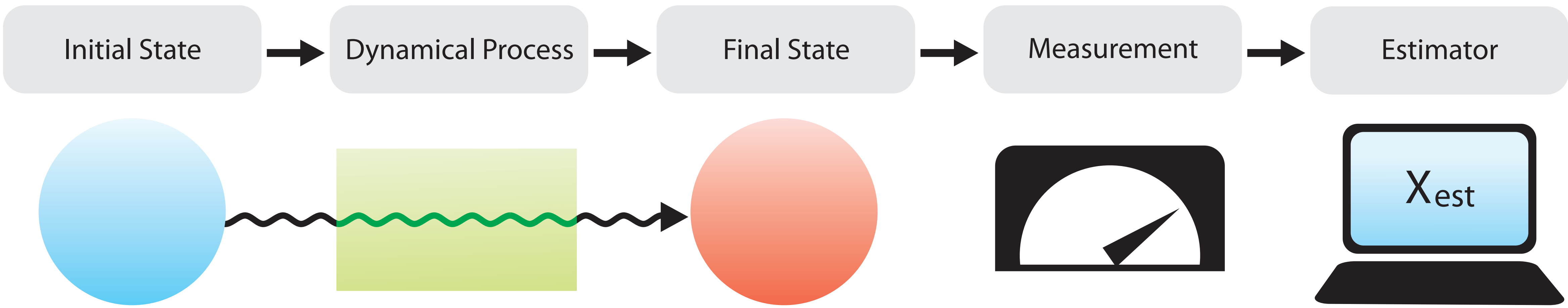
General estimation theory

1. What are the best possible measurements?

2. What are the best incoming states, in order to get better precision?

3. Is it possible to find *general* bounds and strategies for reaching them, which could be applied to many different systems?

Parameter estimation in classical and quantum physics



1. Prepare probe in suitable initial state
2. Send probe through process to be investigated
3. Choose suitable measurement
4. Associate each experimental result j with estimation

$$\delta X \equiv \sqrt{\left\langle \left[X_{\text{est}}(j) - X \right]^2 \right\rangle_j \Big|_{X=X_{\text{true}}}} \quad \rightarrow \quad \text{Merit quantifier}$$

$$\langle X_{\text{est}} \rangle = X_{\text{true}}, \quad d\langle X_{\text{est}} \rangle / dX \Big|_{X=X_{\text{true}}} = 1 \quad \rightarrow \quad \text{Unbiased estimator}$$

Then $\delta X^2 = \Delta^2 X = \left\langle \left[X_{\text{est}} - \langle X_{\text{est}} \rangle \right]^2 \right\rangle \rightarrow$ variance of X_{est} (average is taken over all experimental results)

Estimator depends only on the experimental data.

Classical parameter estimation


H. Cramér


C. R. Rao


R.A. Fisher

Cramér-Rao bound for unbiased estimators:

$$\Delta X \geq 1/\sqrt{N\,F(X)\big|_{X=X_{\text{true}}}}\,,\quad F(X) \equiv \sum_j P_j(X)\left(\frac{d\,\ln\left[P_j(X)\right]}{dX}\right)^2$$

Fisher information

$N \rightarrow$ Number of repetitions of the experiment

$P_j(X) \rightarrow$ probability of getting an experimental result j

or yet, for continuous measurements: $F(X) \equiv \int d\xi\, p(\xi|X)\left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2$

where ξ are the measurement results

(Average over all experimental results)

Derivation of Cramér-Rao relation: See lectures by L. Davidovich at College de France, 2016:

http://www.if.ufrj.br/~ldavid/eng/show_arquivos.php?Id=5

Exercises

1. Show that

$$F(X) \equiv \int d\xi\, p(\xi|X) \left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2 = \int d\xi \frac{1}{p(\xi|X)} \left[\frac{\partial p(\xi|X)}{\partial X}\right]^2$$

$$= 4 \int d\xi \left[\frac{\partial \sqrt{p(\xi|X)}}{\partial X}\right]^2 = -\left\langle \frac{\partial^2}{\partial X^2} \ln p(\xi|X) \right\rangle$$

with similar expressions for a discrete set of measurements. For instance,

$$F(X) = \sum_k \left[\frac{d\sqrt{P_k(X)}}{dX}\right]^2$$

2. Let us consider several identical and independent measurements, so that the probability distribution is $p(\vec{\xi}|X) = p(\xi_1|X) \cdots p(\xi_N|X)$. Show that $F^{(N)}(X) = NF(X)$

Understanding the Fisher information (1)

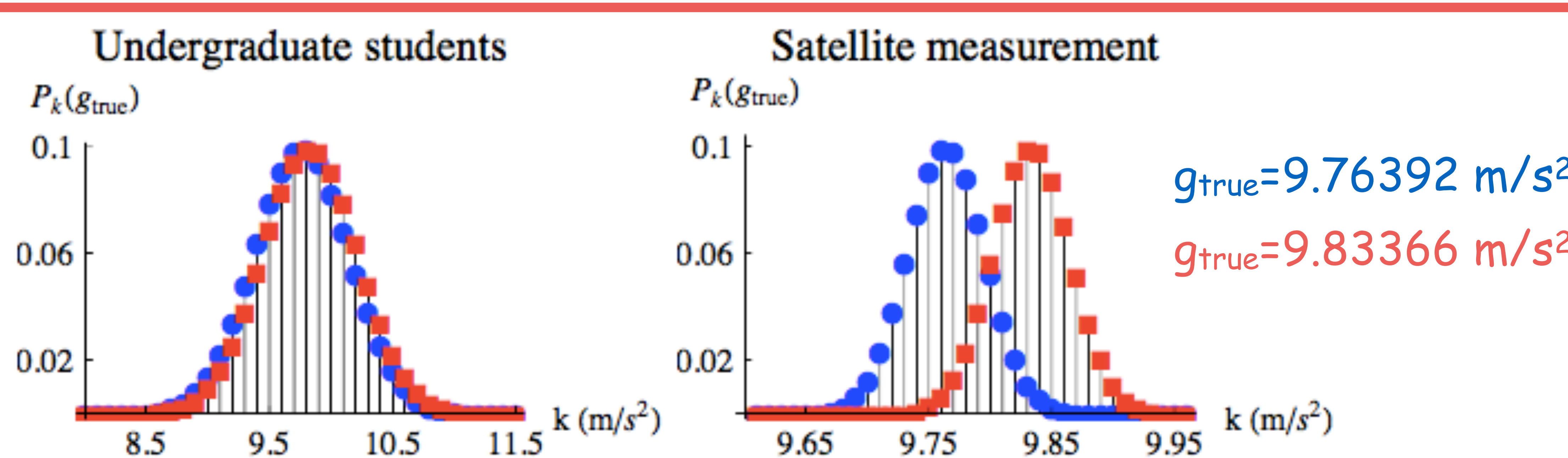
GEOPHYSICAL RESEARCH LETTERS, VOL. 40, 4279–4283, doi:10.1002/grl.50838, 2013

New ultrahigh-resolution picture of Earth's gravity field

Christian Hirt,[1] Sten Claessens,[1] Thomas Fecher,[2] Michael Kuhn,[1] Roland Pail,[2] and Moritz Rexer[1,2]

Márcio Mendes Taddei, Ph. D. thesis, Federal University of Rio de Janeiro, available at arXiv:1407.4343v1 [quant-ph]

The gravitational field is measured by undergraduate students, via an inclined-plane experiment, in two labs, situated at Huáscaran (Peruvian Andes) and the Artic Sea, so g_{true} is different in both cases. Their precision is one decimal place. The same measurement is made by higher-precision satellites, with one additional decimal place.



Values of $P_k(g_{\text{true}})$ for a measurement of g in Huascarán, in the Andes (blue circles) and at the Arctic Sea (red squares). The distributions within each image are different because so is g_{true}. Measurement as made in a simple laboratory (left) is compared to that by higher-precision satellites (right).

Understanding the Fisher information (2)

The higher precision of the satellite experiments implies that it is easier to distinguish the true values of g from the P_k of these measurements. **Important question:** How much does the outcome distribution change by a change of the underlying true value of the parameter? I show now that the Fisher information is a measure of this change.

The distance between two probability distributions $\{P_k\}$ for a given set $\{k\}$ of outcomes, which differ because they belong to two different values x and x' of the parameter, can be defined by the **Hellinger expression** D_H:

$$D_H(x,x') = \sqrt{\frac{1}{2}\sum_k \left[\sqrt{P_k(x)} - \sqrt{P_k(x')}\right]^2}$$

Then,

$$D_H^2(x,x+dx) = \frac{1}{2}\sum_k \left[\sqrt{P_k(x+dx)} - \sqrt{P_k(x)}\right]^2 = \frac{1}{2}\sum_k \left[\frac{d}{dx}\sqrt{P_k(x)}\right]^2 dx^2$$

and

$$\boxed{D_H^2(x,x+dx) = ds_H^2 = \frac{F(x)}{8}dx^2}$$

F(X) as a measure of change of the probability distribution!

Understanding the Fisher information (3)

The expression for the Hellinger distance can be written in terms of the fidelity between the two distributions:

$$D_H(x,x') = \sqrt{\frac{1}{2}\sum_k \left[\sqrt{P_k(x)} - \sqrt{P_k(x')}\right]^2} = \sqrt{1-\sqrt{\Phi_H(x,x')}}$$

where

$$\Phi_H(x,x') = \left[\sum_k \sqrt{P_k(x)P_k(x')}\right]^2$$ (=1 for x=x')

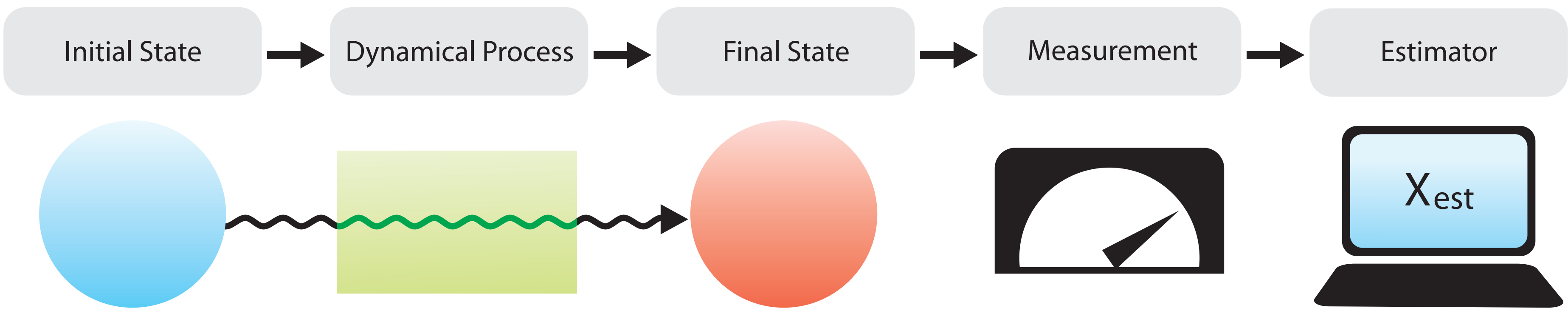
Therefore:

$$\boxed{\Phi_H(x,x') = 1 - \frac{F(x)}{4}dx^2}$$

$$\frac{\sqrt{F(x)}}{2} \rightarrow$$ Speed of change

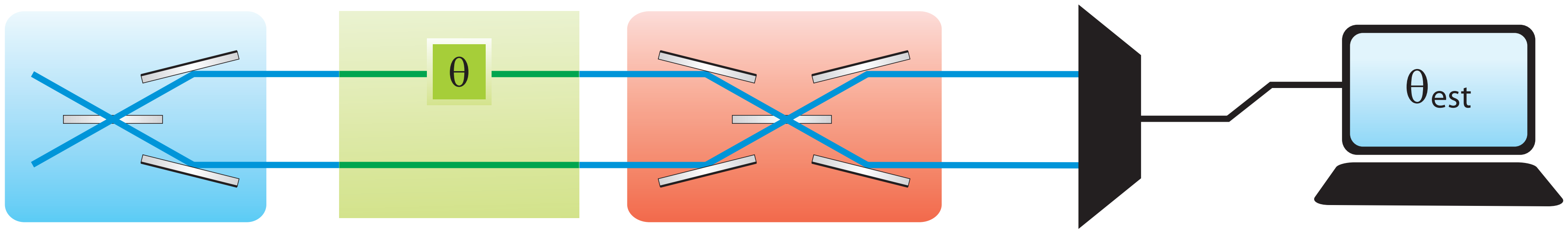
I.2 - Quantum parameter estimation

Quantum parameter estimation



The general idea is the same as before: one sends a probe through a parameter-dependent dynamical process and one measures the final state to determine the parameter. The precision in the determination of the parameter depends now on the distinguishability between quantum states corresponding to nearby values of the parameter.

Example: Optical interferometry



$$\left|\langle\alpha|\alpha e^{i\delta\theta}\rangle\right|^2 = \exp\left(-\left|\alpha\left(1-e^{i\delta\theta}\right)\right|^2\right)$$
$$\approx \exp\left[-\langle n\rangle(\delta\theta)^2\right] \Rightarrow \delta\theta \approx 1/\sqrt{\langle n\rangle}$$

→ Standard limit (shot noise)

Possible method to increase precision for the same average number of photons: Use NOON states [J. J. Bolinguer et al., PRA **54**, R4649 (1996); J. P. Dowling, PRA 57, 4736 (1998)]

$$|\psi(N)\rangle = \left(|N,0\rangle + |0,N\rangle\right)/\sqrt{2} \rightarrow |\psi(N,\theta)\rangle = \left(|N,0\rangle + e^{iN\theta}|0,N\rangle\right)/\sqrt{2}, \quad (\langle n\rangle = N)$$

$$\left|\langle\psi(N)|\psi(N,\delta\theta)\rangle\right|^2 = \cos^2(N\delta\theta/2) \Rightarrow \delta\theta \approx 1/N$$

$$\left[\cos^2(N\delta\theta/2) = 0 \Rightarrow \delta\theta = \pi/N\right]$$

HEISENBERG LIMIT — Precision is better, for the same amount of resources (average number of photons)!

Quantum Fisher Information

(Helstrom, Holevo, Braunstein and Caves)

$$F\left(X;\{\hat{E}_\xi\}\right) \equiv \int d\xi\, p(\xi \mid X)\left(\frac{d\,\ln\left[p(\xi \mid X)\right]}{dX}\right)^2$$

$$p(\xi \mid X) = \mathrm{Tr}\left[\hat{\rho}(X)\hat{E}_\xi\right]$$

$$\int d\xi \hat{E}_\xi = \hat{1}$$

POVM

This corresponds to a given quantum measurement. Ultimate lower bound for $\langle(\Delta X_{\rm est})^2\rangle$: optimize over all quantum measurements so that

$$\mathscr{F}_Q(X) = \max_{\{E_\xi\}} F\left(X;\left\{E_\xi\right\}\right)$$

Quantum Fisher Information

Quantum Fisher information for pure states

(See notes for derivation)

Initial state of the probe: $|\psi(0)\rangle$

Final X-dependent state: $|\psi(X)\rangle = \hat{U}(X)|\psi(0)\rangle$, $\hat{U}(X)$ unitary operator.

Then (Helstrom 1976):

$$\boxed{\mathcal{F}_Q(X) = 4\langle(\Delta\hat{H})^2\rangle_0\,, \quad \langle(\Delta\hat{H})^2\rangle_0 \equiv \langle\psi(0)|\left[\hat{H}(X) - \langle\hat{H}(X)\rangle_0\right]^2|\psi(0)\rangle}$$

where

$$\boxed{\hat{H}(X) \equiv i\frac{d\hat{U}^\dagger(X)}{dX}\hat{U}(X)}$$

If $\hat{U}(X) = \exp(i\hat{O}X)$, $\hat{O}$ independent of X, then $\hat{H} = \hat{O}$

$$\boxed{\delta x \geq 1/2\sqrt{\nu\left\langle\Delta\hat{H}^2\right\rangle}}$$ $\Rightarrow$ Should maximize the variance to get better precision!

Another expression for the quantum Fisher information

From

$$\mathcal{F}_Q(X) = 4\langle(\Delta\hat{H})^2\rangle_0\,, \quad \langle(\Delta\hat{H})^2\rangle_0 \equiv \langle\psi(0)|\left[\hat{H}(X) - \langle\hat{H}(X)\rangle_0\right]^2|\psi(0)\rangle$$

and $\hat{H}(X) \equiv i\frac{d\hat{U}^\dagger(X)}{dX}\hat{U}(X)$

it follows that

$$\boxed{\mathcal{F}_Q(X) = 4\left[\frac{d\langle\psi(X)|}{dX}\frac{d|\psi(X)\rangle}{dX} - \left|\frac{d\langle\psi(X)|}{dX}|\psi(X)\rangle\right|^2\right]}$$

Exercise: Show this!

Geometrical interpretation of the quantum Fisher information

Remember that, for classical probability distributions, one had

$$\Phi_H(x,x') = \left[\sum_k \sqrt{P_k(x)P_k(x')}\right]^2, \qquad \Phi_H(x,x') = 1 - \frac{F(x)}{4}dx^2$$

Using the expressions of the probabilities in terms of $\hat{E}_k$, the Bures fidelity between two density operators $\hat{\rho}$ and $\hat{\sigma}$ is defined as

$$\Phi_B(\hat{\rho},\hat{\sigma}) = \min_{\{\hat{E}_k\}} \left[\sum_k \sqrt{\mathrm{Tr}(\hat{\rho}\hat{E}_k)\mathrm{Tr}(\hat{\sigma}\hat{E}_k)}\right]^2 = \min_{\{\hat{E}_k\}} \left[\sum_k \sqrt{P_k(\hat{\rho})P_k(\hat{\sigma})}\right]^2$$

This can be shown to be equal to: $\boxed{\Phi_B(\hat{\rho}_1,\hat{\rho}_2) \equiv \left(\mathrm{Tr}\sqrt{\hat{\rho}_1^{1/2}\hat{\rho}_2\hat{\rho}_1^{1/2}}\right)^2}$

Minimization of Φ_H leads to maximization of F(x), thus yielding the quantum Fisher information.

$$\boxed{\sqrt{\mathscr{F}_Q}/2 \rightarrow \text{speed}}$$

Bures' Fidelity: $\Phi_B(\hat{\rho}_1,\hat{\rho}_2) \equiv \left(\mathrm{Tr}\sqrt{\hat{\rho}_1^{1/2}\hat{\rho}_2\hat{\rho}_1^{1/2}}\right)^2 = \left|\langle\psi_1|\psi_2\rangle\right|^2$ (pure states)

$$\Rightarrow \Phi_B\left[\hat{\rho}(X),\hat{\rho}(X+\delta X)\right] = 1-(\delta X)^2\mathscr{F}_Q\left[\hat{\rho}(X)\right]/4 + O\left[(\delta X)^4\right]$$

Example 1: Optical interferometry

$\hat{n} = \hat{a}^\dagger a \to$ Generator of phase displacements $|\alpha\rangle \to |\alpha \exp(i\theta)\rangle$

$\Rightarrow \mathcal{F}_Q(\theta) = 4\langle(\Delta\hat{n})^2\rangle_0$ where $\langle(\Delta\hat{n})^2\rangle_0$ is the photon-number variance in the upper arm.

$\Rightarrow \delta\theta \geq \dfrac{1}{2\sqrt{\langle(\Delta\hat{n})^2\rangle}} \quad (\nu = 1)$ $\quad \nu \to$ Number of repetitions

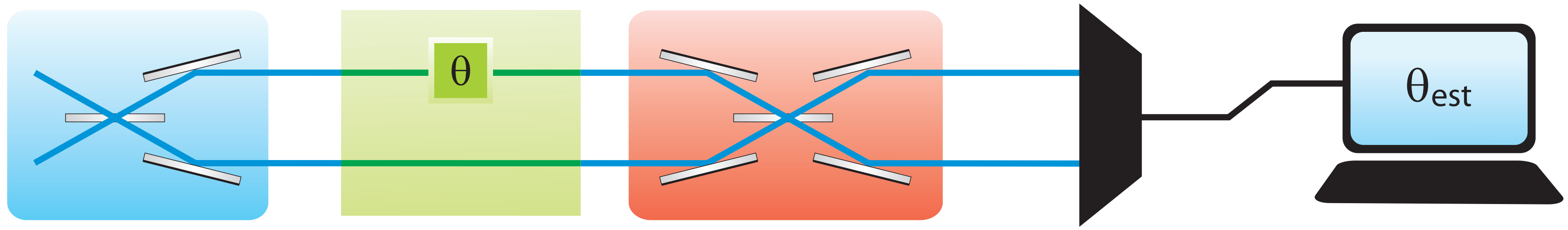
Standard limit: coherent states

$$\mathcal{F}_Q(\theta) = 4\langle(\Delta\hat{n})^2\rangle_0 = 4\langle\hat{n}\rangle \Rightarrow \delta\theta \geq \frac{1}{2\sqrt{\langle n\rangle}}$$

This lower bound is better by a factor of two than the bound found before, which was $\delta\theta_{\min} = 1/\sqrt{\langle n\rangle}$. This earlier bound corresponds to comparing the displaced-phase coherent state in the upper arm of an interferometer with an undisplaced coherent state with the same amplitude in the other arm.

The result found here indicates that a better measurement of the phase is possible: indeed, a homodyne measurement allows the comparison of the displaced coherent state with a classical reference field (local oscillator), which is just a coherent state with a number of photons much larger than that of the measured state — this yields a better precision in the estimation of the phase.

Example 1: Optical interferometry



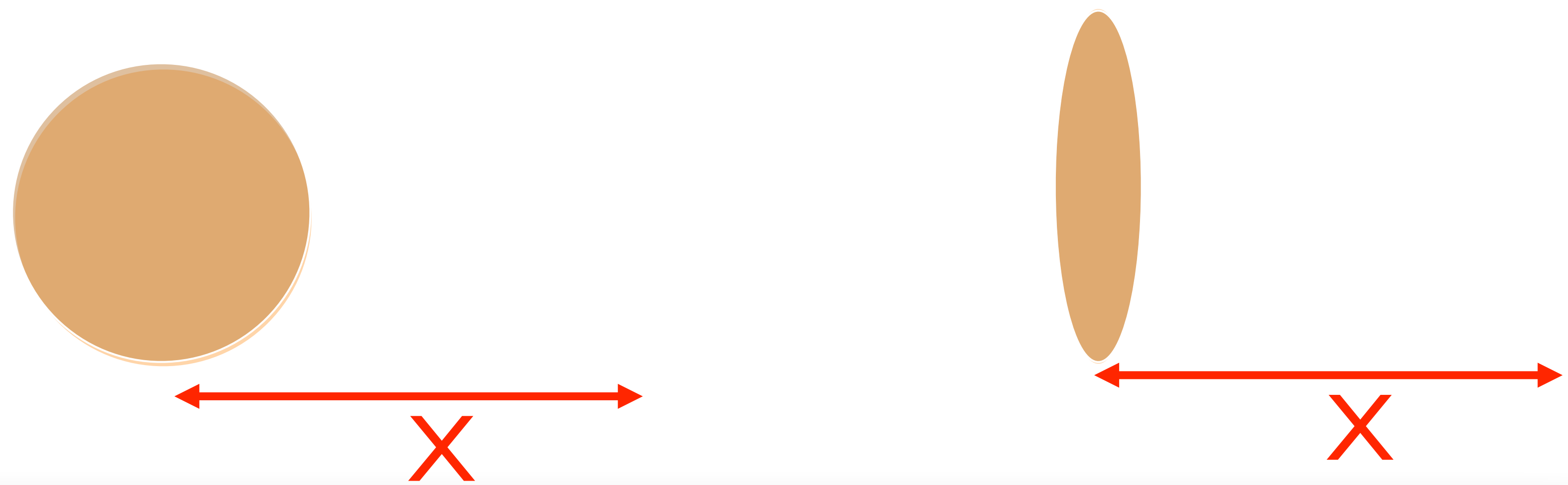
Increasing the precision: maximize variance with NOON states:

$$|\psi(N)\rangle = \left(|N,0\rangle + |0,N\rangle\right)/\sqrt{2}$$ —> entangled state

$$\mathcal{F}_Q(\theta) = 4\langle(\Delta\hat{n})^2\rangle_0 \Rightarrow \delta\theta \geq \frac{1}{2\sqrt{\langle(\Delta\hat{n})^2\rangle}} \quad (\nu = 1)$$

$$\boxed{\left\langle(\Delta\hat{n})^2\right\rangle_0 = \frac{N^2}{4} \Rightarrow \delta\theta \geq \frac{1}{N}}$$

Example 2: Spatial displacement



$$|\psi(X)\rangle = e^{iX\hat{P}}|\psi(0)\rangle \Rightarrow \hat{H} = i\frac{d\hat{U}^\dagger}{dX}\hat{U}(X) = \hat{P}$$

$$\mathcal{F}_Q(X) = 4\langle(\Delta\hat{P})^2\rangle_0 \Rightarrow \langle(\Delta X)^2\rangle \geq \frac{1}{4\langle(\Delta\hat{P})^2\rangle}$$

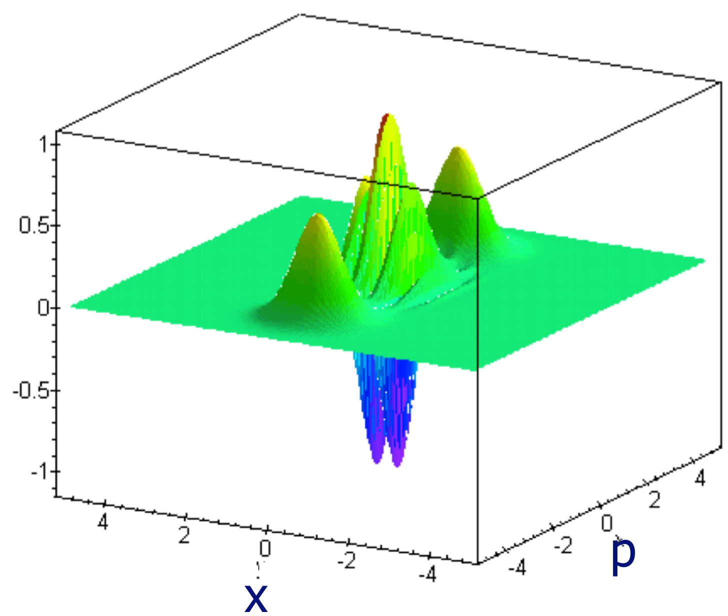
Coherent state: $\langle(\Delta\hat{P})^2\rangle_0 = 1/2 \Rightarrow \langle(\Delta X)^2\rangle = 1/2$ —> standard quantum limit — coherent state saturates Cramér-Rao bound

Maximizing variance of P for better precision: e.g., squeezed states —> Also saturate the bound (Gaussian states)

Looks like Heisenberg uncertainty relation, but X is a parameter, not an operator!

Example 3: Phase-space displacement

A sensitive instrument…

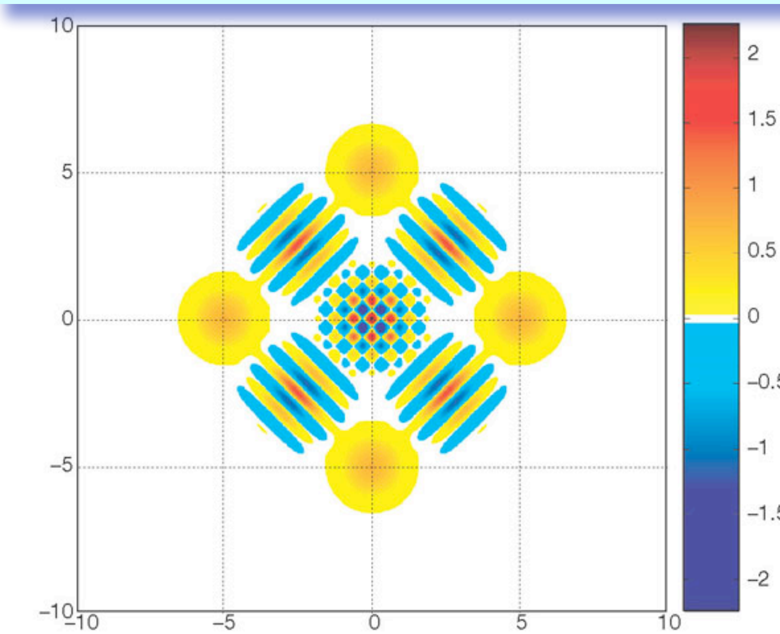


$$|\psi\rangle = \mathrm{N}\left(|\alpha\rangle + |-\alpha\rangle\right)$$

$$\Delta X \approx \frac{1}{|\alpha|}$$

Sub-Planck sensitivity

W. Zurek, Nature **412**, 712 (2001)



Vlastakis et al., Science **342**, 607 (2013)

$$|\psi\rangle = \mathrm{N}'\left(|\alpha\rangle + |-\alpha\rangle + |i\alpha\rangle + |-i\alpha\rangle\right)$$

PHYSICAL REVIEW A **73**, 023803 (2006)

Sub-Planck phase-space structures and Heisenberg-limited measurements

F. Toscano,[1] D. A. R. Dalvit,[2] L. Davidovich,[1] and W. H. Zurek[2]

PHYSICAL REVIEW A **94**, 022313 (2016)

Measurement of a microwave field amplitude beyond the standard quantum limit

M. Penasa,[1] S. Gerlich,[1] T. Rybarczyk,[1] V. Métillon,[1] M. Brune,[1] J. M. Raimond,[1] S. Haroche,[1] L. Davidovich,[2] and I. Dotsenko[1,*]

Possible strategies for quantum-enhanced metrology (1)

Single probe

Recall that $\mathcal{F}_Q(|\psi\rangle) = 4\langle(\Delta\hat{H})^2\rangle$ so in order to increase the precision one needs to choose a state $|\psi\rangle$ that maximizes the variance $\langle(\Delta\hat{H})^2\rangle$. If $\hat{H}$ has a discrete and bounded spectrum, this is accomplished by letting

$$|\psi\rangle_{\text{opt}} = \frac{1}{\sqrt{2}}\left(|\lambda_{\text{max}}\rangle + |\lambda_{\text{min}}\rangle\right)$$

where $|\lambda_{\text{max}}\rangle$ and $|\lambda_{\text{min}}\rangle$ are eigenstates of $\hat{H}$ corresponding to the maximum and minimum eigenvalues.

Then $\langle(\Delta\hat{H})^2\rangle = (\lambda_{\text{max}} - \lambda_{\text{min}})^2/4$ and

$$\Delta\varphi_{(1)} \geq \frac{1}{\sqrt{\nu}\left(\lambda_{\text{max}} - \lambda_{\text{min}}\right)}$$

(ν —> number of repetitions of single probe experiment)

Question: What is the best strategy if one has N probes?

Entanglement-assisted parameter estimation: phase estimation

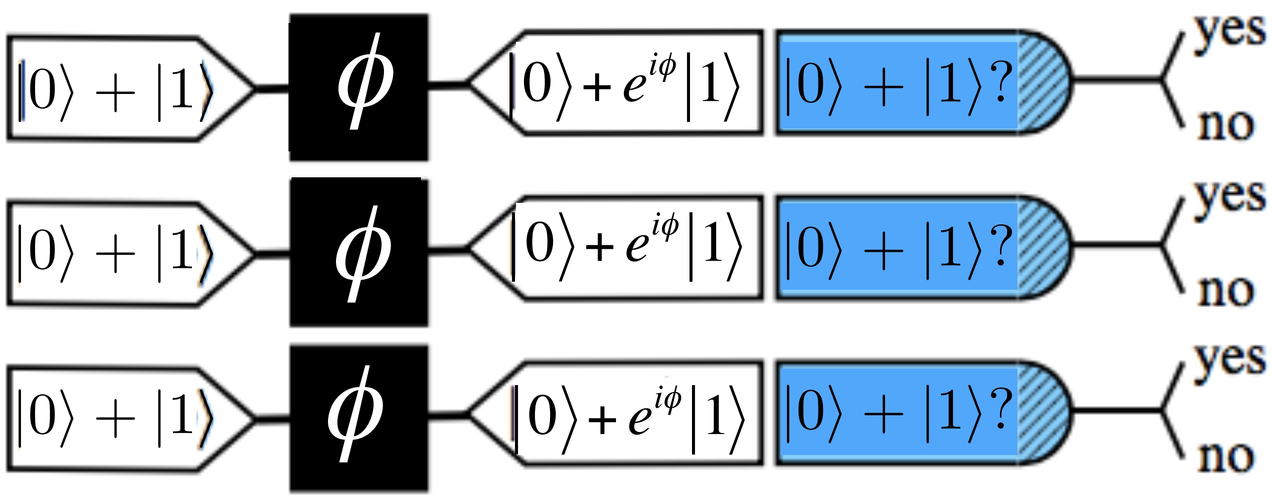
The problem. One wants to estimate a small change of phase between states of a two-level system, which would allow to estimate say a small electromagnetic field, or yet a transition frequency between the two states.

Two possible strategies:

[Figures adapted from V. Giovannetti, S. Lloyd and L. Maccone, Nature Photonics **5**, 222–229 (2011)]

Separable

$$\left(|0\rangle+|1\rangle\right)\rightarrow \exp\left[i\left(1+\hat{\sigma}_z\right)\phi/2\right]\left(|0\rangle+|1\rangle\right)$$



$$p_S\left(\text{yes}\right)\equiv p_S=\left(1+\cos\phi\right)/2$$

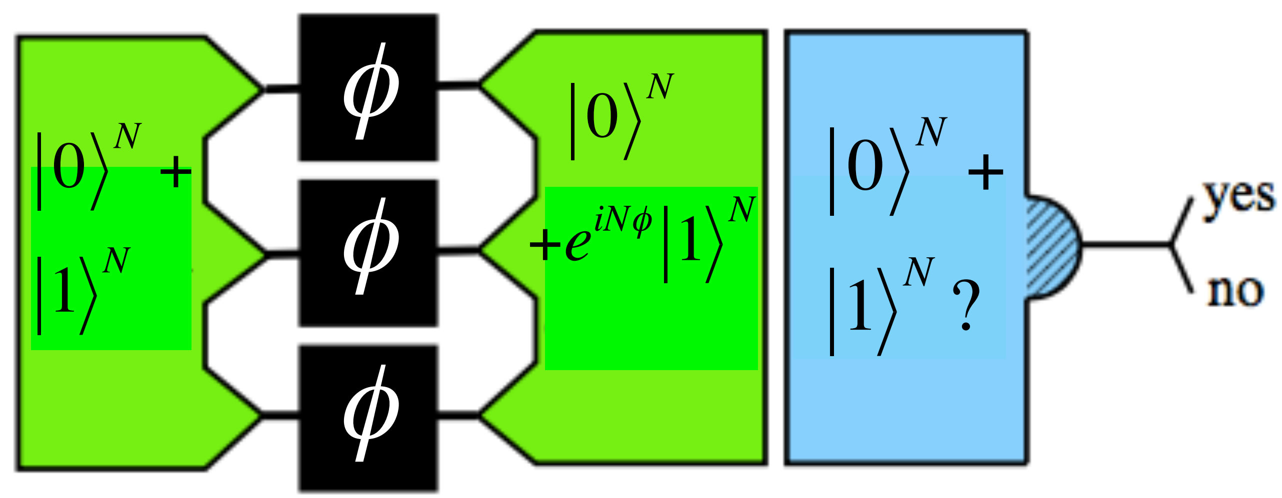
$$p_S\left(\text{no}\right)=1-p_S=\left(1-\cos\phi\right)/2$$

$$F_S\left(\phi\right)=\left(\frac{1}{p_S}+\frac{1}{1-p_S}\right)\left[\frac{\partial p_s}{\partial\phi}\right]^2=\left[\frac{1}{p_S\left(1-p_S\right)}\right]\left[\frac{\partial p_s}{\partial\phi}\right]^2$$

$$=1$$

$$\delta\phi_S\geq 1/\sqrt{NF_S\left(\phi\right)}=1/\sqrt{N}$$

Entangled



$$p_E\left(\text{yes}\right)\equiv p_E=\left(1+\cos N\phi\right)/2$$

$$p_E\left(\text{no}\right)=1-p_E=\left(1-\cos N\phi\right)/2$$

$$F_E\left(\phi\right)=\left[\frac{1}{p_E\left(1-p_E\right)}\right]\left[\frac{\partial p_E}{\partial\phi}\right]^2=1$$

$$\delta\phi_E\geq 1/\sqrt{NF_E\left(\phi\right)}=1/N$$

Entanglement-assisted parameter estimation: phase estimation (2)

Are these the best measurements?

$$\left(|0\rangle+|1\rangle\right)\rightarrow \exp\left[i\left(1+\hat{\sigma}_z\right)\phi/2\right]\left(|0\rangle+|1\rangle\right)$$

1. Separable qubits.

We know that for the best measurement $\mathcal{F}_Q(\phi) = 4\langle(\Delta\hat{H})^2\rangle_0$, where $\hat{H}$ here is the generator of phase displacements: $\hat{H} = (1+\hat{\sigma}_z)/2$. Since for the initial state $|+\rangle$ we have $\langle(\Delta\hat{H})^2\rangle_0 = 1/4$, it follows that the measurement of $\hat{\sigma}_x$ maximizes the Fisher information, leading to the corresponding Cramér-Rao bound in $\delta\phi \geq 1/\sqrt{N\mathcal{F}_Q(\phi)} = 1/\sqrt{N}$, the so-called standard limit.

2. Entangled qubits.

The generator of phase displacements is $\hat{H} = \sum_{i=1}^{N}\left(1+\hat{\sigma}_z^{(i)}\right)/2$, so that $\langle\psi(0)|(\Delta\hat{H})^2|\psi(0)\rangle = N^2/4$, which means that the above measurement leads to the maximum value of the Fisher information and to the Cramér-Rao bound in $\delta\phi \geq 1/\sqrt{\mathcal{F}_Q(\phi)} = 1/N$, the Heisenberg limit.

Entanglement-assisted parameter estimation: phase estimation (3)

2. Entangled qubits.

Bound can be achieved with local measurements! Measure observable $\hat{\sigma}^{\otimes N} = \hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_x^{(2)} \cdots \otimes \hat{\sigma}_x^{(N)}$ on final state $|0\rangle^N + e^{iN\phi}|1\rangle^N$

Get

$$\left\langle \hat{\sigma}^{\otimes N} \right\rangle = \cos(N\varphi)$$

$$\Delta \hat{\sigma}^{\otimes N} = \left|\sin(N\varphi)\right|$$

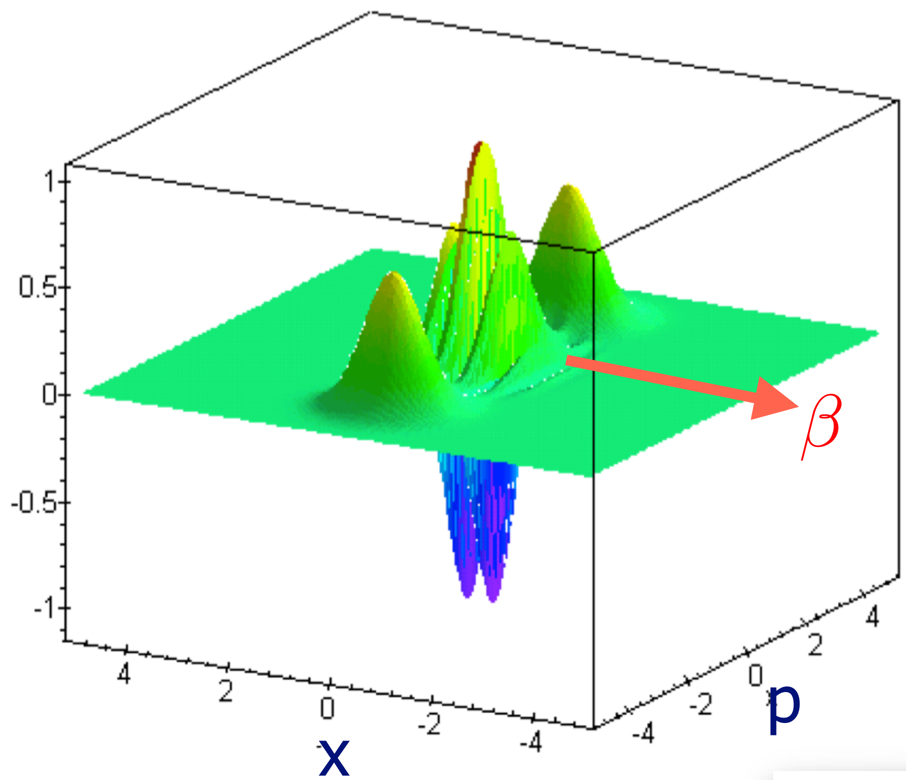
So, from error propagation:

$$\delta\varphi = \frac{\Delta \hat{\sigma}^{\otimes N}}{\partial \left\langle \hat{\sigma}^{\otimes N} \right\rangle / \partial\varphi} = \frac{1}{N}$$

which coincides with the Heisenberg bound.

Therefore, only the initial entanglement counts!

EXPERIMENT 2: Measuring sub-Planck state displacements in phase space



$$\beta \approx \frac{1}{|\alpha|} \Rightarrow \text{orthogonality}$$

$$|\text{cat}\rangle = (|\alpha\rangle + |-\alpha\rangle)/\sqrt{2}$$

PHYSICAL REVIEW A **73**, 023803 (2006)

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Looking for a classical-like distribution in phase space

We look for a distribution in phase space with the following property:

$$\int dp W(q,p) = \langle q|\hat{\rho}|q\rangle, \ \int dq W(q,p) = \langle p|\hat{\rho}|p\rangle$$

Pure state:

$$\langle q|\hat{\rho}|q\rangle = |\psi(q)|^2, \ \langle p|\hat{\rho}|p\rangle = |\tilde{\psi}(p)|^2$$

Property should be valid with rotated axes:

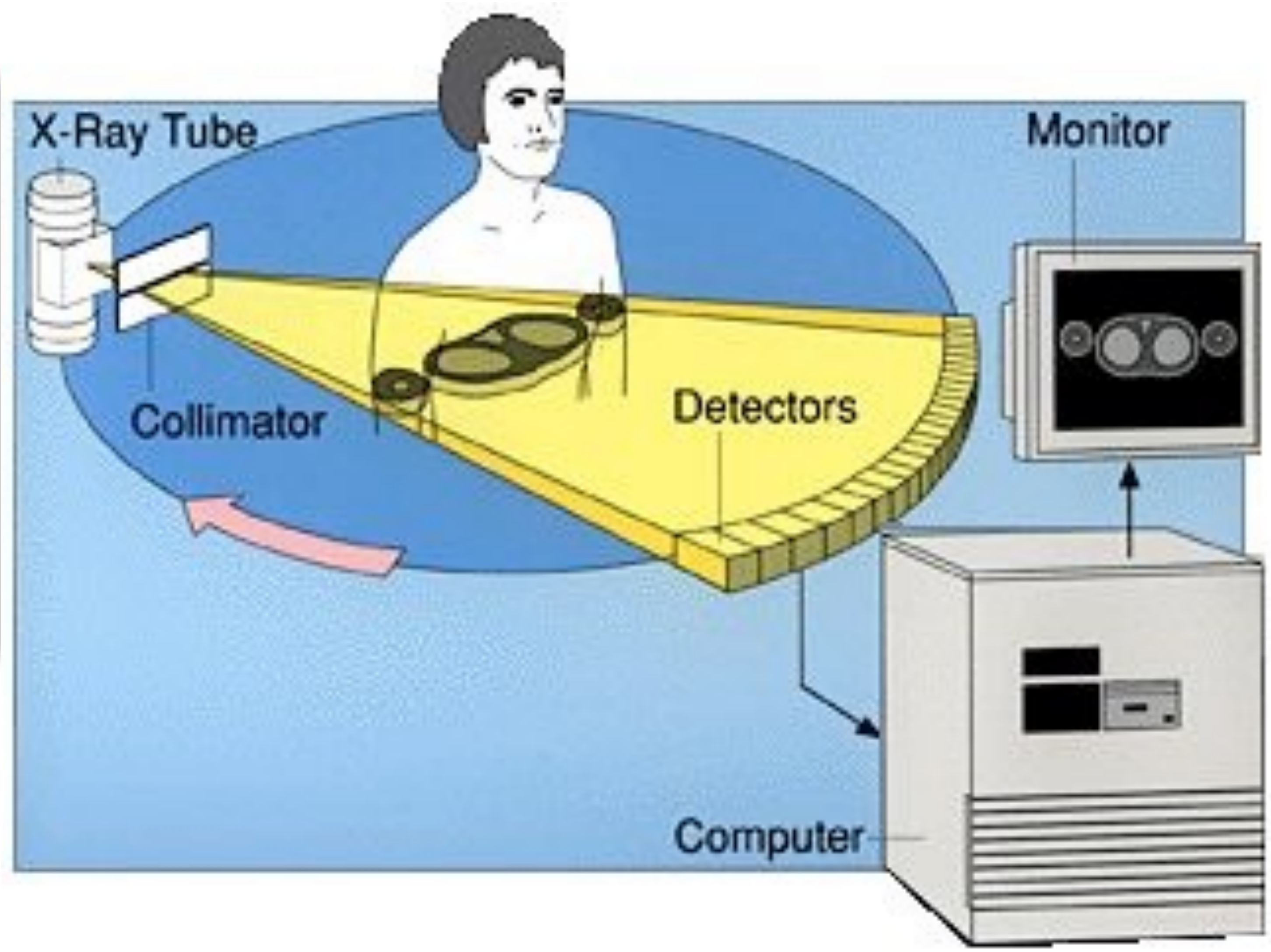
$$\int W(q_\theta \cos\theta - p_\theta \sin\theta, q_\theta \sin\theta + p_\theta \cos\theta) dp_\theta$$

$$= P(q_\theta) = \langle q|\hat{U}^\dagger(\theta)\hat{\rho}\hat{U}(\theta)|q\rangle$$

RADON TRANSFORM (1917)

$P(q_\theta)$ determines uniquely $W(q,p)$! $\rightarrow$ inverse Radon transform

$\rightarrow$ tomography



Cormack and Hounsfield: Nobel Prize in Medicine (1979)

Quantum mechanics: $P(q_\theta)$

$\Rightarrow$ Wigner distribution

(Bertrand and Bertrand, 1987)

Wigner distribution

Wigner, 1932: Quantum corrections to classical statistical mechanics

$$\hat{x}|x\rangle = x|x\rangle$$

$$W(x,p) = \frac{1}{\pi\hbar}\int \langle x+x'|\hat{\rho}|x-x'\rangle e^{-2ipx'/\hbar}dx'$$

Moyal, 1949: Average of operators in symmetric form

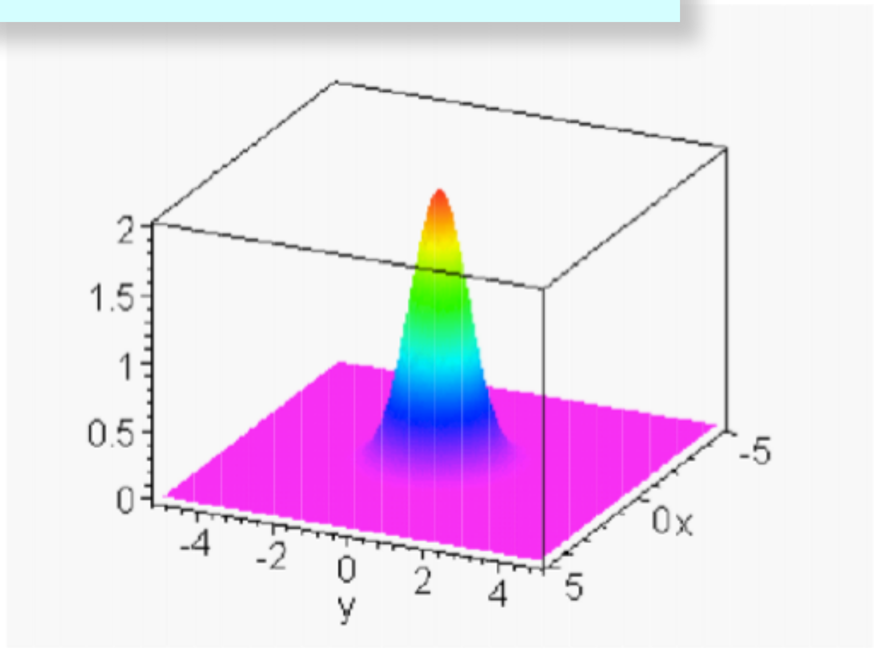
$$Tr\left[\hat{\rho}\left(\hat{x}\hat{p}+\hat{p}\hat{x}\right)/2\right] = \int dxdpW(x,p)xp$$

Density matrix from W:

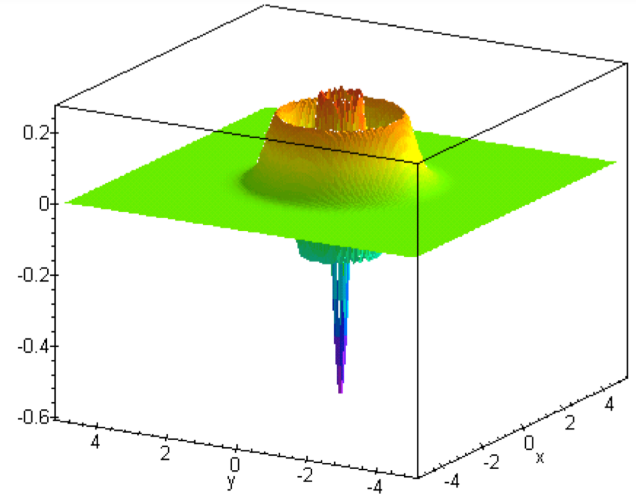
$$\langle x+x'|\hat{\rho}|x-x'\rangle = \int W(x,p)e^{2ipx'/\hbar}dp/\hbar$$

Examples of Wigner distributions for harmonic oscillator

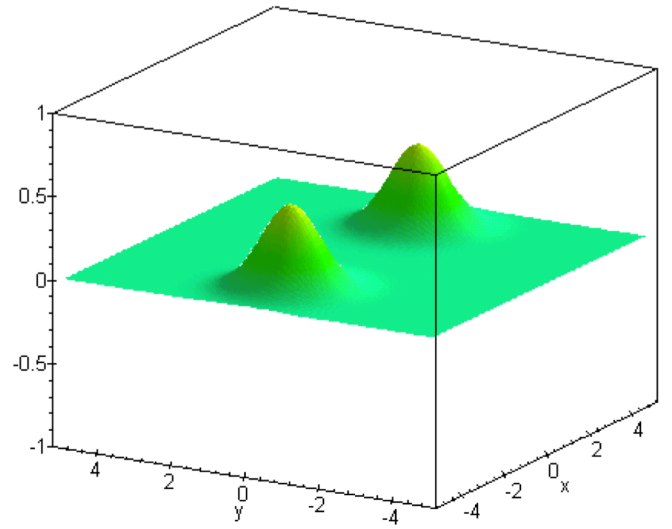
Ground state



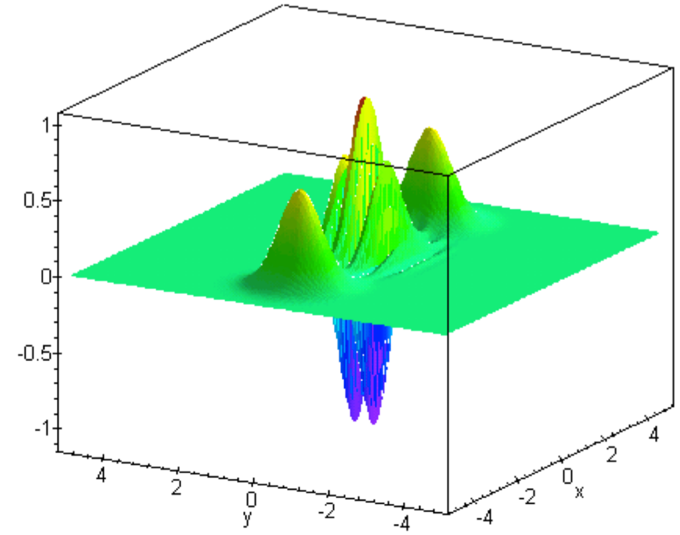
Fock state with n=3



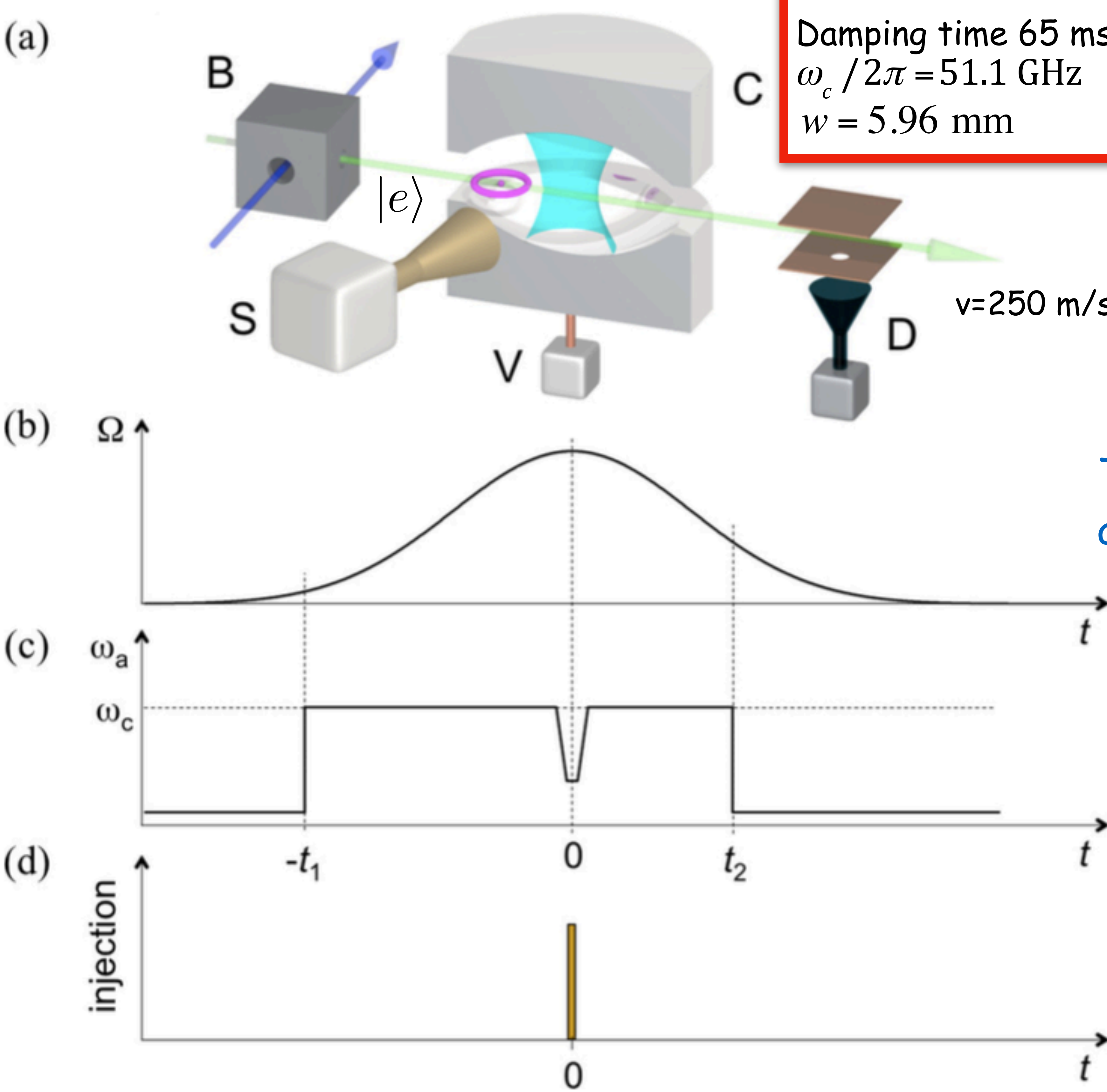
Mixed state $(|\alpha\rangle\langle\alpha|+|-\alpha\rangle\langle-\alpha|)/2$



Superposition $\propto\ |\alpha\rangle+|-\alpha\rangle$



Experimental procedure



Damping time 65 ms
$\omega_c / 2\pi = 51.1$ GHz
$w = 5.96$ mm

Coherent state
with 12.7 photons
$\{|g\rangle, |e\rangle\} \rightarrow n = 50, 51$
$\Omega_0 / 2\pi = 46$ kHz
$T_{\max} \simeq 42\ \mu s$

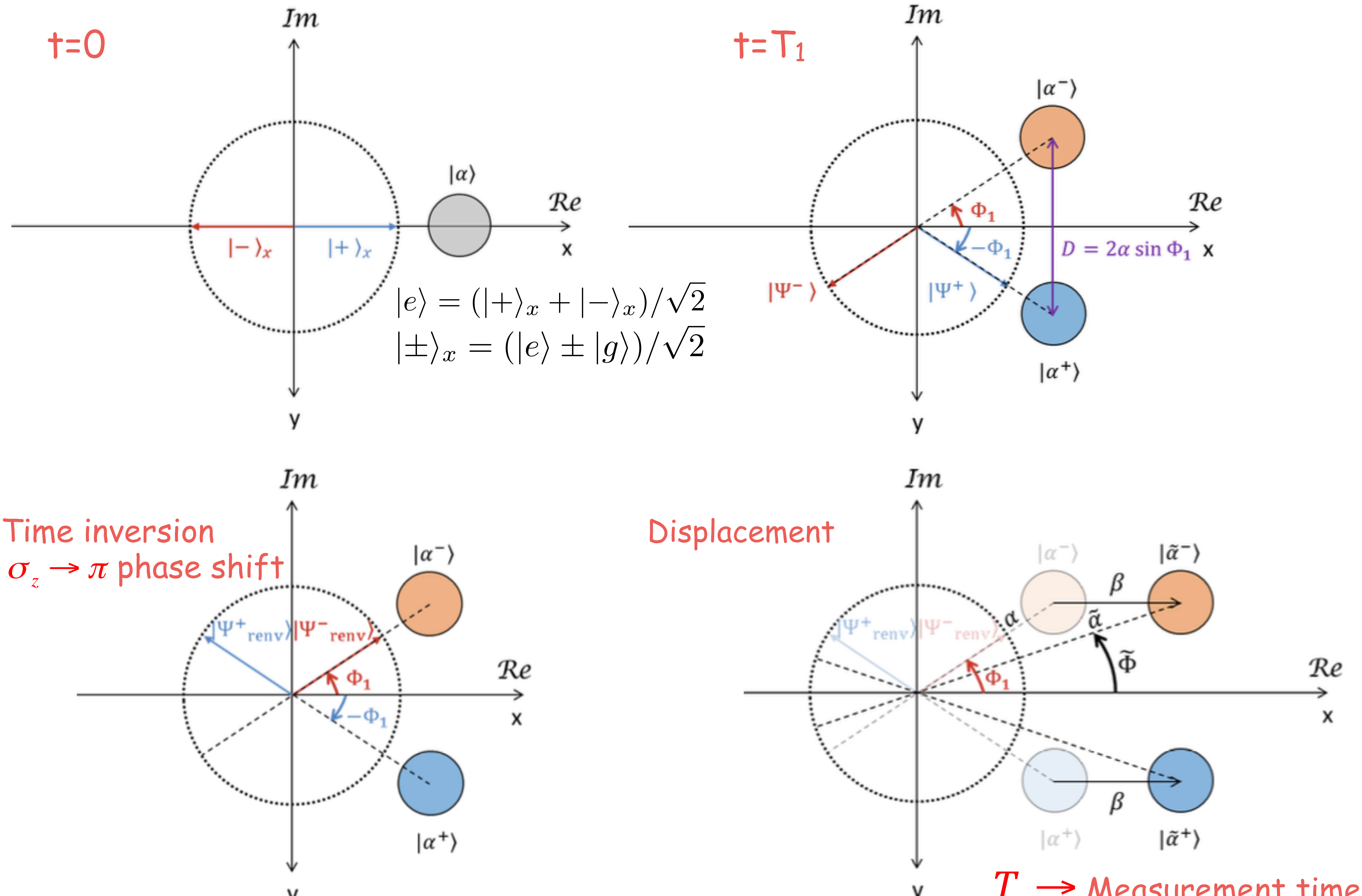
v=250 m/s

Temporal variation of the atom-cavity coupling

Modulation of atomic frequency $(\sigma_z \rightarrow \pi)$

Field β to be measured is injected into the cavity at t=0

Measurement protocol



t=0

$$|e\rangle = (|+\rangle_x + |-\rangle_x)/\sqrt{2}$$
$$|\pm\rangle_x = (|e\rangle \pm |g\rangle)/\sqrt{2}$$

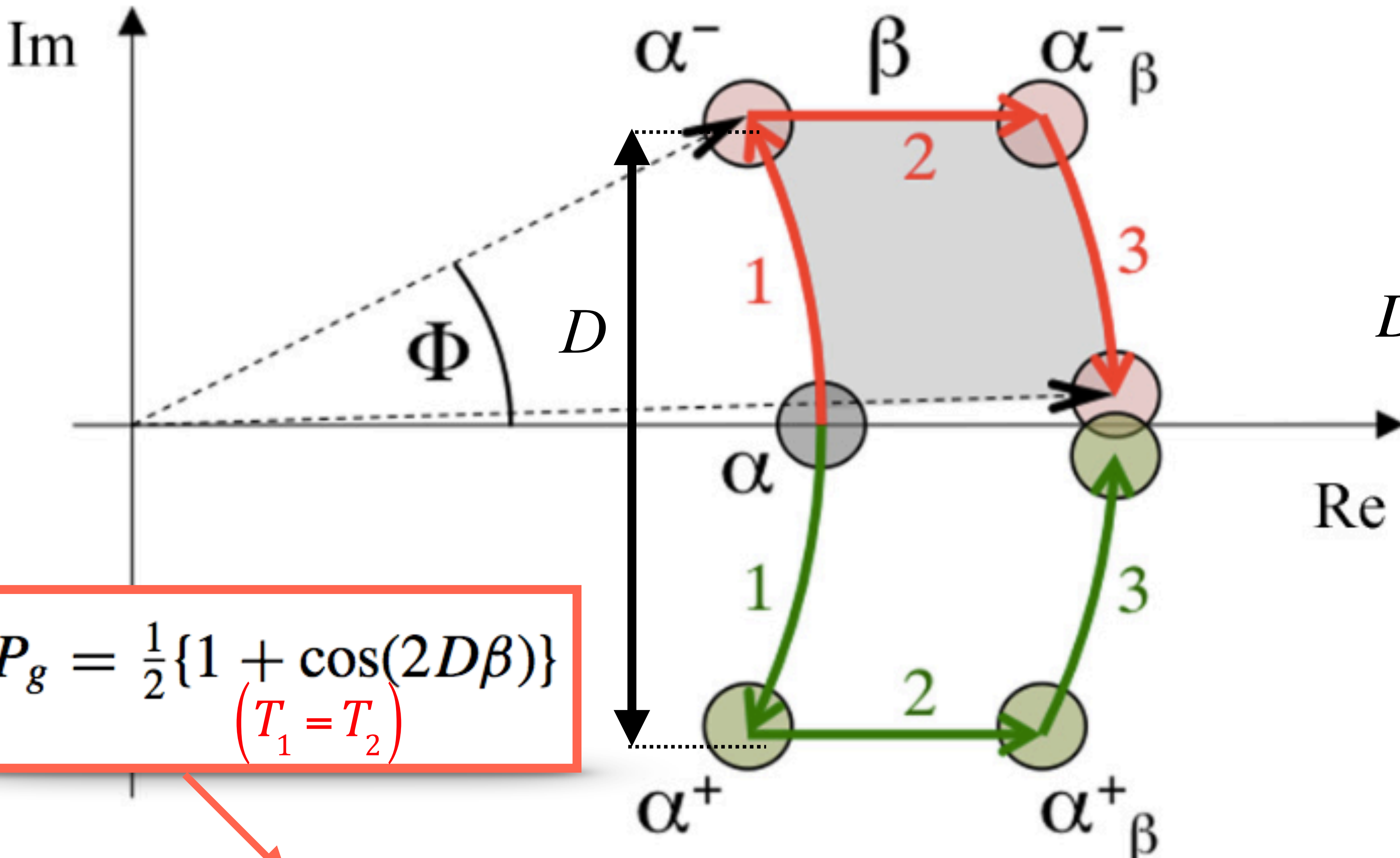
t=T$_1$

Time inversion
$\sigma_z \rightarrow \pi$ phase shift

Displacement

$T_2 \rightarrow$ Measurement time

Measurement protocol



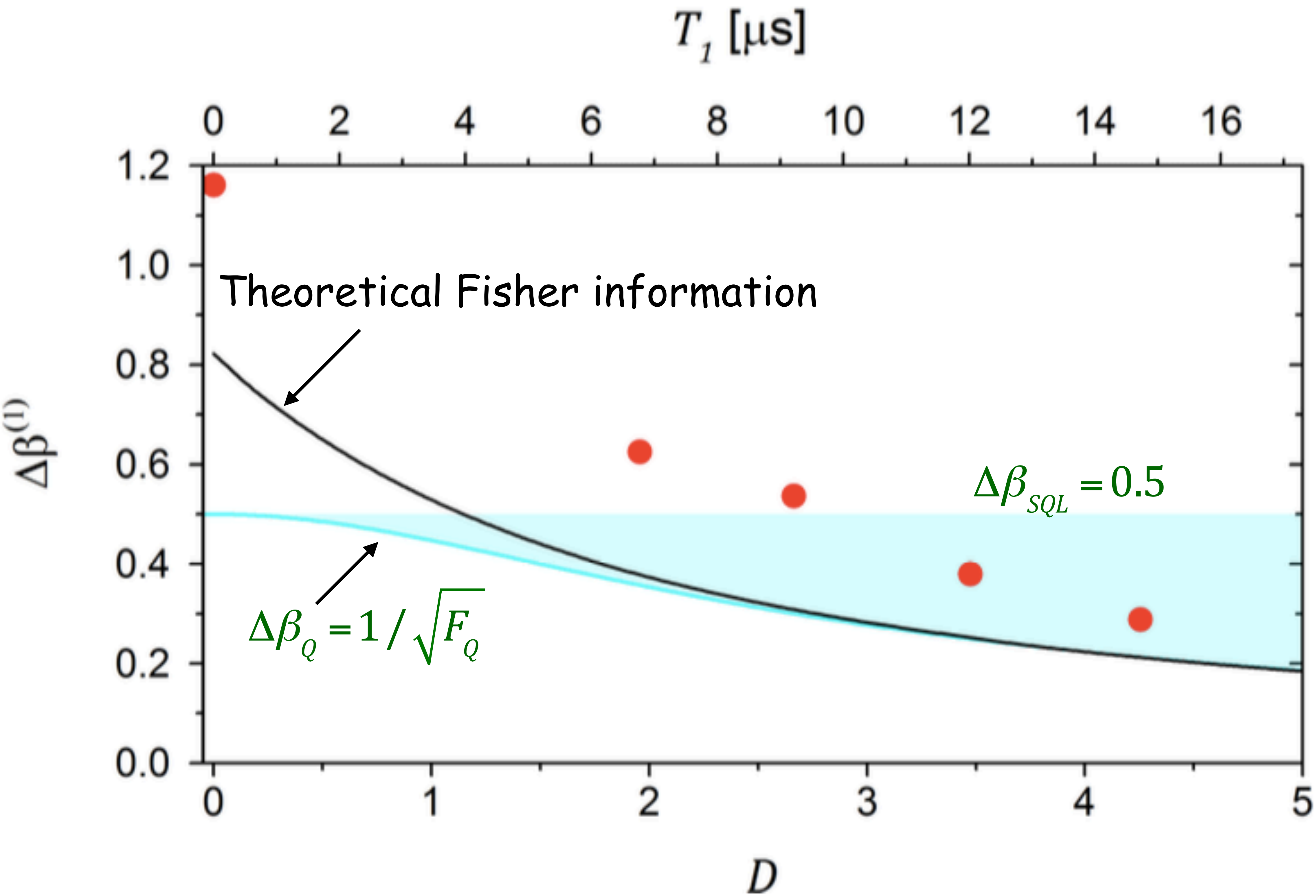
$D = 2\alpha \sin\Phi$

$$P_g = \frac{1}{2}\{1 + \cos(2D\beta)\}$$

$(T_1 = T_2)$

$$F(\beta) \equiv \sum_{g,e} P_j(\beta)\left(\frac{d \ln\left[P_j(\beta)\right]}{d\beta}\right)^2 \Rightarrow \Delta\beta \geq 1/\sqrt{\nu F(\beta)}$$

Experimental results



Best result: F_{exp} = 3SQL

$$10\log_{10}\left(\sqrt{F_{\exp}/F_{SQL}}\right) \approx 2{,}4 \text{ dB}$$