Quantum Science and Technology - Argentina 2019

XXI GIAMBIAGI WINTER SCHOOL

Quantum simulations and quantum metrology with cold trapped ions – July 15-24

Instituto de Física UFRJ

Towards the ultimate precision limits in parameter estimation: An introduction to quantum metrology

Luiz Davidovich
Instituto de Física - Universidade Federal do Rio de Janeiro

Outline of the lectures

These three lectures will focus on recent developments in quantum metrology. The main questions to be answered are:

(i) What are the ultimate precision limits in the estimation of parameters, according to classical mechanics and quantum mechanics?
(ii) Are there fundamental limits? Is quantum mechanics helpful in reaching better precision?
(iii) How to cope with the deleterious effects of noise?

Our discussion is restricted to local quantum metrology: in this case, one is not interested in an optimal globally-valid estimation strategy, valid for any value of the parameter to be estimated, but one wants instead to estimate a parameter confined to some small range. The techniques to be developed are useful, for instance, for estimating parameters that undergo small changes around a known value, like sensing phase changes in gravitational-wave detectors or yet very small forces or magnetic fields — These are typical quantum sensing problems

Summary of the lectures

The lectures will be organized as follows:

LECTURE 1. Examples of metrological tasks. Quantum metrology and optical interferometers. Shot-noise and Heisenberg limits. Radiation pressure in gravitational-wave interferometers. Classical bounds on precision: The Cramér-Rao bound and introduction of the Fisher information.

LECTURE 2. Extension of Cramér-Rao bound and Fisher information to quantum mechanics. Quantum Fisher information for noiseless systems. The role of entanglement. Application to atomic interferometry. Beyond the standard quantum limit: experimental results with optical interferometers and cavity QED.

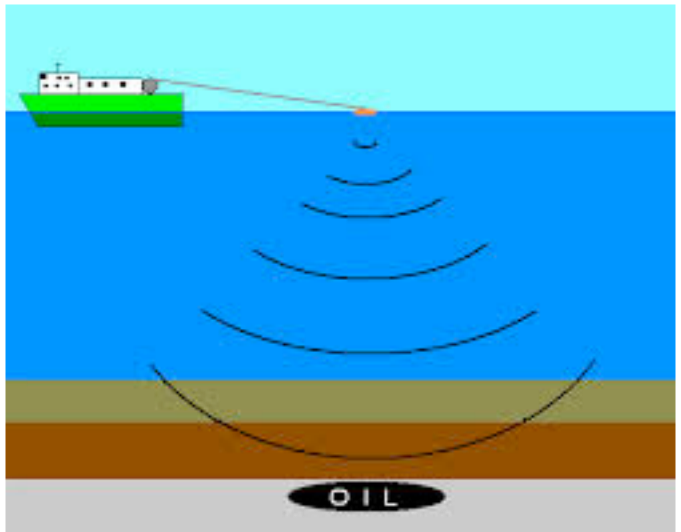
LECTURE 3. Noisy quantum-enhanced metrology: General framework for evaluating the ultimate precision limit in the estimation of parameters. Quantum channels. Application to optical interferometers. Quantum metrology and the energy-time uncertainty relation. Application to atomic decay and dephasing.

For more details, see Lectures at College de France (2016):
http://www.if.ufrj.br/~ldavid/eng/show_arquivos.php?Id=5

I.1 - General introduction: parameter estimation and classical limits on precision

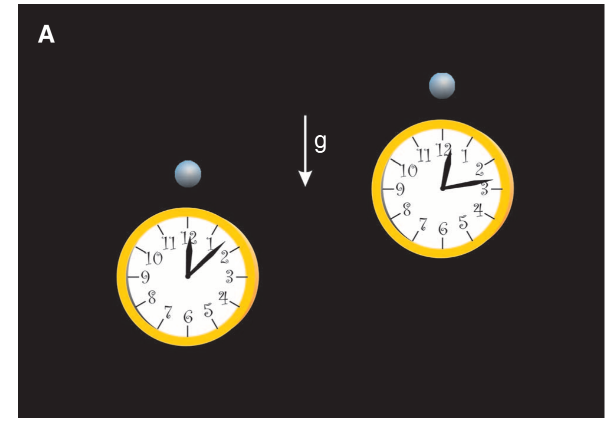
Parameter estimation

Depth of an oil well



Time duration of a process

Transition frequency



$$\Delta h = 33\,\mathrm{cm}$$

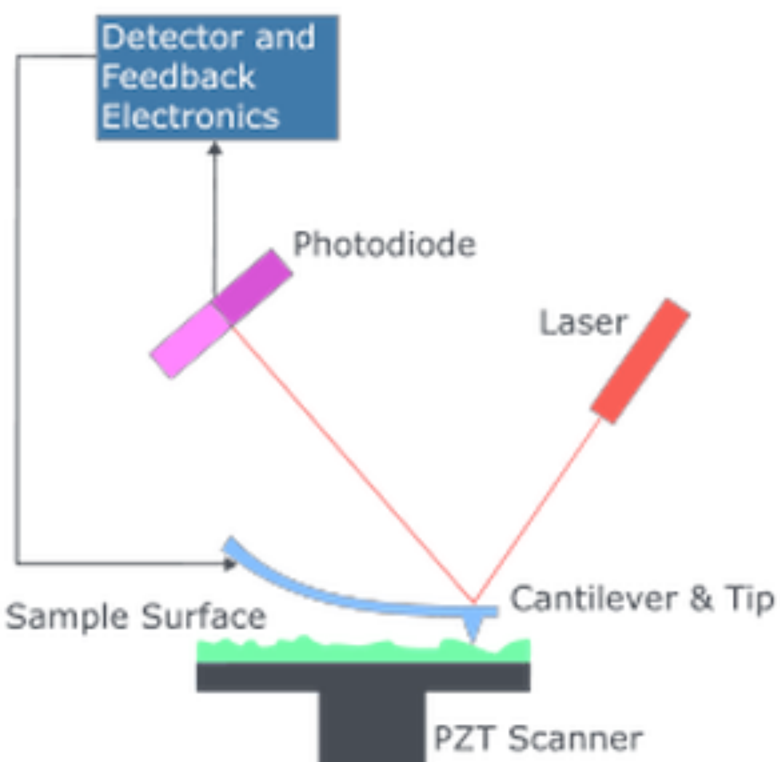
$$\frac{\Delta f}{f} = (4.1 \pm 1.6) \times 10^{-17}$$

Optical Clocks and Relativity

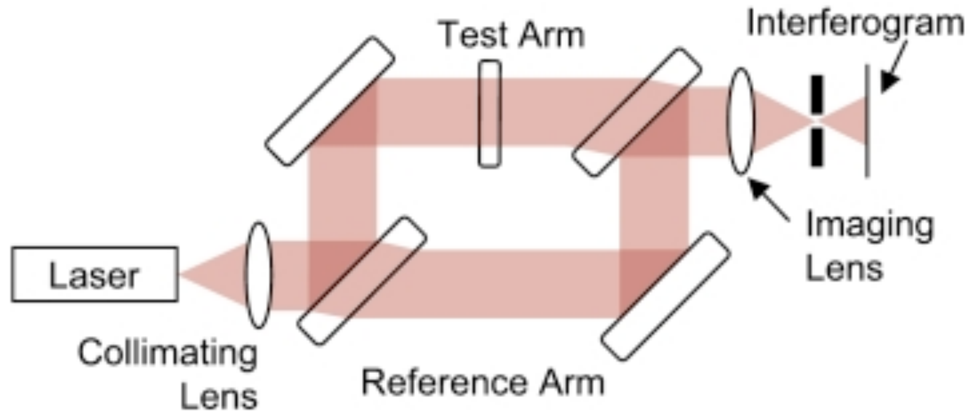
C. W. Chou,* D. B. Hume, T. Rosenband, D. J. Wineland

24 SEPTEMBER 2010 VOL 329 **SCIENCE**

Weak forces or small displacements



Phase displacements in interferometers



Laser Interferometer Gravitational Wave Observatory

nature photonics
LETTERS
PUBLISHED ONLINE: 21 JULY 2013 | DOI: 10.1038/NPHOTON.2013.177

Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light

The LIGO Scientific Collaboration*

Experiments: Parameter estimation beyond classical physics in the XXI century

Phase resolution

Letter

Nature Photonics **4**, 357 - 360 (2010)
Published online: 4 April 2010 | doi:10.1038/nphoton.2010.39

Experimental quantum-enhanced estimation of a lossy phase shift

M. Kacprowicz[1], R. Demkowicz-Dobrzański[1,2], W. Wasilewski[2], K. Banaszek[1,2] & I. A. Walmsley[3]

NATURE PHOTONICS | LETTER

Entanglement-enhanced measurement of a completely unknown optical phase

G. Y. Xiang, B. L. Higgins, D. W. Berry, H. M. Wiseman & G. J. Pryde

Letters to Nature

Nature **429**, 161-164 (13 May 2004) | doi:10.1038/nature02493; Received 22 December 2003; Accepted 16 March 2004

Super-resolving phase measurements with a multiphoton entangled state

M. W. Mitchell, J. S. Lundeen & A. M. Steinberg

1. Department of Physics, University of Toronto, 60 St George Street, Toronto, Ontario M5S 1A7, Canada

New Journal of Physics
The open access journal at the forefront of physics

Beating the standard quantum limit: phase super-sensitivity of *N*-photon interferometers

Ryo Okamoto[1,2,5], Holger F Hofmann[3], Tomohisa Nagata[1], Jeremy L O'Brien[4], Keiji Sasaki[1] and Shigeki Takeuchi[1,2]

Science 4 May 2007:
Vol. 316 no. 5825 pp. 726-729
DOI: 10.1126/science.1138007

REPORT

Beating the Standard Quantum Limit with Four-Entangled Photons

Tomohisa Nagata[1], Ryo Okamoto[1,2], Jeremy L. O'Brien[3,4], Keiji Sasaki[1], Shigeki Takeuchi[1,2,*]

Experiments: Parameter estimation beyond classical physics in the XXI century

Atomic clocks

10960–10965 | PNAS | July 7, 2009 | vol. 106 | no. 27

Mesoscopic atomic entanglement for precision measurements beyond the standard quantum limit

J. Appel, P. J. Windpassinger, D. Oblak, U. B. Hoff, N. Kjærgaard, and E. S. Polzik[1]

Danish National Research Foundation Center for Quantum Optics, The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Letter

Nature **443**, 316-319 (21 September 2006) | doi:10.1038/nature05101; Received 5 May 2006; Accepted 18 July 2006

'Designer atoms' for quantum metrology

C. F. Roos[1,2], M. Chwalla[1], K. Kim[1], M. Riebe[1] & R. Blatt[1,2]

Science 4 June 2004:
Vol. 304 no. 5676 pp. 1476–1478
DOI: 10.1126/science.1097576

REPORT

Toward Heisenberg-Limited Spectroscopy with Multiparticle Entangled States

D. Leibfried*, M. D. Barrett†, T. Schaetz, J. Britton, J. Chiaverini, W. M. Itano, J. D. Jost, C. Langer, D. J. Wineland

Physics ABOUT BROWSE JOURNALISTS

Focus: Atomic Clock Beats the Quantum Limit

June 25, 2010 • *Phys. Rev. Focus* 25, 24

Researchers beat the quantum-mechanical fluctuations in an atomic clock by linking many atoms into an entangled quantum state and pushing the fluctuations into a realm that doesn't influence the time measurement.

New Journal of Physics
The open-access journal for physics

Entanglement-assisted atomic clock beyond the projection noise limit

Anne Louchet-Chauvet[1], Jürgen Appel, Jelmer J Renema, Daniel Oblak, Niels Kjaergaard[2] and Eugene S Polzik[3]

Implementation of Cavity Squeezing of a Collective Atomic Spin

Ian D. Leroux, Monika H. Schleier-Smith, and Vladan Vuletić

Phys. Rev. Lett. **104**, 073602 – Published 17 February 2010; Erratum Phys. Rev. Lett. **106**, 129902 (2011)

Experiments: Parameter estimation beyond classical physics in the XXI century

Magnetometers

NATURE | LETTER

Interaction-based quantum metrology showing scaling beyond the Heisenberg limit

M. Napolitano, M. Koschorreck, B. Dubost, N. Behbood, R. J. Sewell & M. W. Mitchell

NATURE | LETTER

Nature **510**, 376–380 (19 June 2014)

日本語要約

Measurement of the magnetic interaction between two bound electrons of two separate ions

Shlomi Kotler, Nitzan Akerman, Nir Navon, Yinnon Glickman & Roee Ozeri

Magnetic Sensitivity Beyond the Projection Noise Limit by Spin Squeezing

R. J. Sewell, M. Koschorreck, M. Napolitano, B. Dubost, N. Behbood, and M. W. Mitchell
Phys. Rev. Lett. **109**, 253605 – Published 19 December 2012

Quantum Noise Limited and Entanglement-Assisted Magnetometry

W. Wasilewski, K. Jensen, H. Krauter, J. J. Renema, M. V. Balabas, and E. S. Polzik
Phys. Rev. Lett. **104**, 133601 – Published 31 March 2010; Erratum Phys. Rev. Lett. **104**, 209902 (2010)

Increasing Sensing Resolution with Error Correction

G. Arrad, Y. Vinkler, D. Aharonov, and A. Retzker
Phys. Rev. Lett. **112**, 150801 – Published 16 April 2014

Quantum Error Correction for Metrology

E. M. Kessler, I. Lovchinsky, A. O. Sushkov, and M. D. Lukin
Phys. Rev. Lett. **112**, 150802 – Published 16 April 2014

REPORTS

29 MAY 2009 VOL 324 SCIENCE

Magnetic Field Sensing Beyond the Standard Quantum Limit Using 10-Spin NOON States

Jonathan A. Jones,[1] Steven D. Karlen,[2] Joseph Fitzsimons,[2,3] Arzhang Ardavan,[1] Simon C. Benjamin,[2,4] G. Andrew D. Briggs,[2] John J. L. Morton[1,2]*

Experiments: Parameter estimation beyond classical physics in the XXI century

Force, displacement, and tilt

nature
International journal of science
NEWS · 15 FEBRUARY 2019

Gravitational-wave observatory LIGO set to double its detecting power

A planned US$35-million upgrade could enable LIGO to spot one black-hole merger per hour by the mid-2020s.

Davide Castelvecchi

nature physics
LETTERS
Published: 27 May 2019
https://doi.org/10.1038/s41567-019-0533-5

Continuous force and displacement measurement below the standard quantum limit

David Mason[1,2,3], Junxin Chen[1,2,3], Massimiliano Rossi[1,2,3], Yeghishe Tsaturyan[1] and Albert Schliesser[1,2*]

nature COMMUNICATIONS
Published: 02 July 2019
ARTICLE
https://doi.org/10.1038/s41467-019-10576-4 OPEN

Motional Fock states for quantum-enhanced amplitude and phase measurements with trapped ions

Fabian Wolf[1], Chunyan Shi[1], Jan C. Heip[1], Manuel Gessner[2,5], Luca Pezzè[2], Augusto Smerzi[2,4], Marius Schulte[3], Klemens Hammerer[3] & Piet O. Schmidt[1,4]

PHYSICAL REVIEW A **00**, 000300(R) (2018)
Rapid Communications

Quantum-enhanced sensing from hyperentanglement

S. P. Walborn, A. H. Pimentel, L. Davidovich, and R. L. de Matos Filho
Instituto de Física, Universidade Federal do Rio de Janeiro, P.O. Box 68528, Rio de Janeiro, RJ 21941-972, Brazil

Letter
Vol. 42, No. 13 / July 1 2017 / *Optics Letters* 2479
Optics Letters

Ultrasensitive inverse weak-value tilt meter

Julián Martínez-Rincón,[1,2,*] Christopher A. Mullarkey,[1,2] Gerardo I. Viza,[1,2,7] Wei-Tao Liu,[1,2,3,4] and John C. Howell[1,2,5,6]

PHYSICAL REVIEW A **94**, 022313 (2016)

Measurement of a microwave field amplitude beyond the standard quantum limit

M. Penasa,[1] S. Gerlich,[1] T. Rybarczyk,[1] V. Métillon,[1] M. Brune,[1] J. M. Raimond,[1] S. Haroche,[1] L. Davidovich,[2] and I. Dotsenko[1,*]

Parameter estimation and uncertainty relations

What is the meaning of

★ Time-energy uncertainty relation?

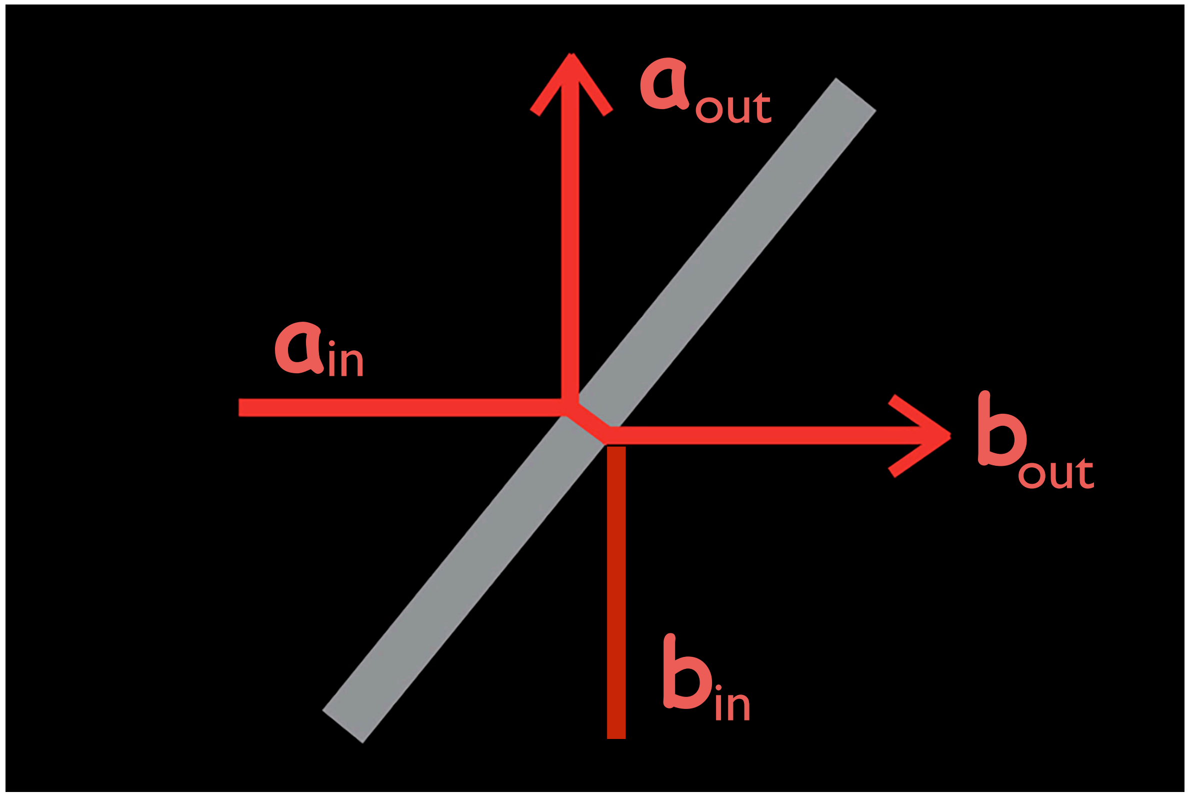
$$\Delta E \Delta T \geq \hbar / 2$$

★ Number-phase uncertainty relation?

$$\Delta N \Delta \phi \geq \hbar / 2$$

We shall see that quantum parameter estimation allows to understand these relations in terms of uncertainties in the estimation of parameters: while Heisenberg uncertainty relations are associated with Hermitian operators, the theory of parameter estimation allows one to obtain uncertainty relations for parameters, like time or phase, with no need to associate them to suitable Hermitian operators.

Photons and beam splitters I



$$\begin{pmatrix} a_{out} \\ b_{out} \end{pmatrix} = \begin{pmatrix} r & t \\ t & r \end{pmatrix} \begin{pmatrix} a_{in} \\ b_{in} \end{pmatrix}$$

$$|a_{out}|^2 + |b_{out}|^2 = |a_{in}|^2 + |b_{in}|^2 \Rightarrow$$

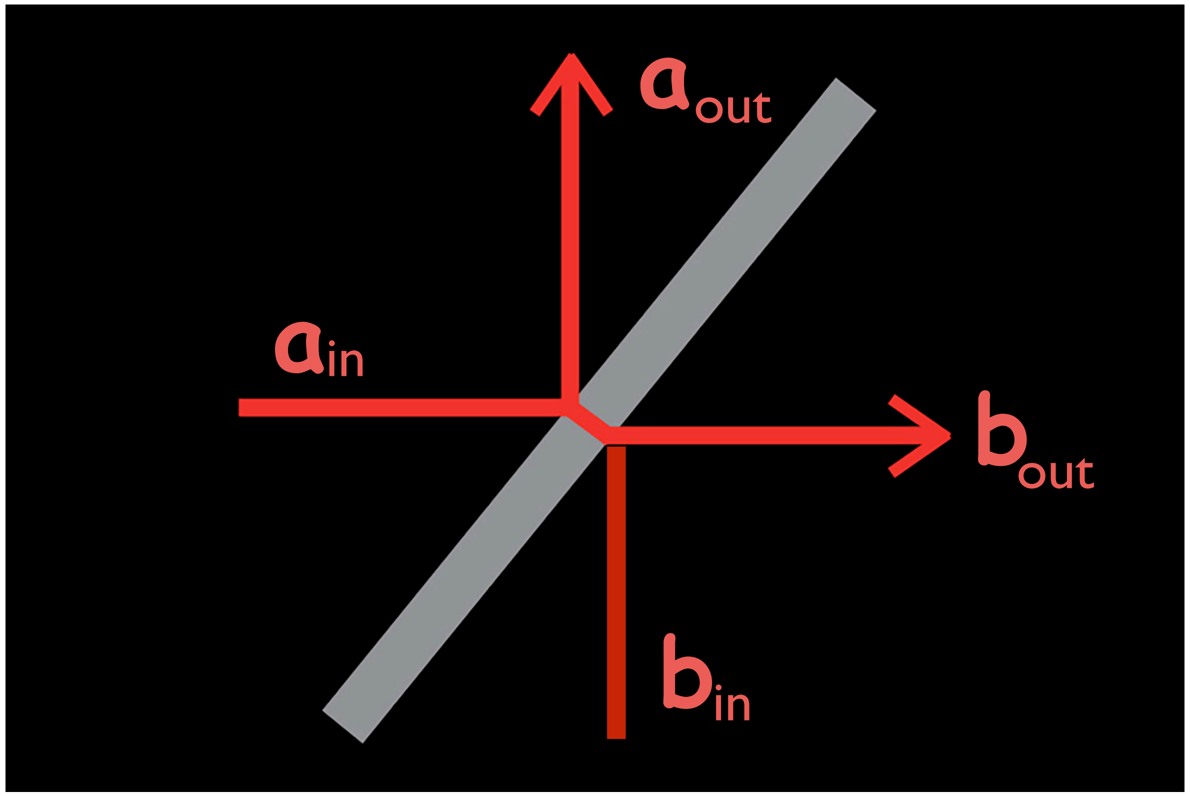
$$|r|^2 + |t|^2 = 1,\ rt^* + r^*t = 0$$

Balanced interferometer:

$$r = \frac{1}{\sqrt{2}},\ t = \frac{i}{\sqrt{2}}$$

$$\begin{pmatrix} a_{out} \\ b_{out} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} a_{in} \\ b_{in} \end{pmatrix}$$

Photons and beam splitters II



Heisenberg picture!

Same for operators:

$$\begin{pmatrix} \hat{a}_{out} \\ \hat{b}_{out} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_{in} \\ \hat{b}_{in} \end{pmatrix}$$

$$\left[\hat{a}_{in}, \hat{a}_{in}^{\dagger}\right] = 1, \ \left[\hat{b}_{in}, \hat{b}_{in}^{\dagger}\right] = 1, \ \left[\hat{a}_{in}, \hat{b}_{in}^{\dagger}\right] = 0, \ \left[\hat{a}_{in}, \hat{b}_{in}\right] = 0$$

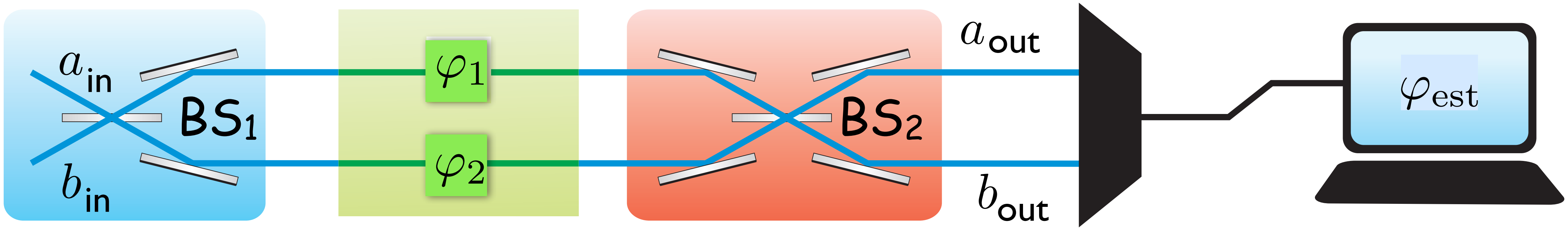
EXERCISE 1: Show that this transformation preserves number of photons and commutation relations

Corresponding evolution operator:

$$\hat{U} = \exp\left[-\frac{i\pi}{4}\left(\hat{a}\hat{b}^{\dagger} + \hat{a}^{\dagger}\hat{b}\right)\right] \Rightarrow \begin{aligned} \hat{U}^{\dagger}\hat{a}_{in}\hat{U} &= \frac{1}{\sqrt{2}}\left(\hat{a}_{in} + i\hat{b}_{in}\right) = \hat{a}_{out} \\ \hat{U}^{\dagger}\hat{b}_{in}\hat{U} &= \frac{1}{\sqrt{2}}\left(i\hat{a}_{in} + \hat{b}_{in}\right) = \hat{b}_{out} \end{aligned}$$

EXERCISE 2: Show this.

An example: optical interferometry

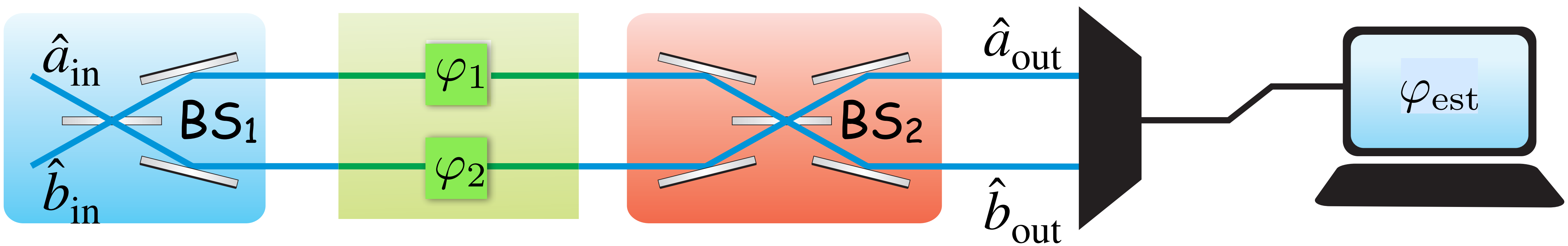


Mach-Zender interferometer: a beam with complex amplitude **a_{in}** is split on a balanced beam splitter BS_1 and the two resulting beams acquire phases φ_1 and φ_2, interfering on the second beam splitter BS_2. The photon numbers $n_{a_{out}}$ and $n_{b_{out}}$ are measured at the output ports. One could also have two incident beams, with complex amplitudes **a_{in}** and **b_{in}**.

The outgoing fields are related to the incoming ones through the transformation (note that **a_{out}**=**a_{in}**, **b_{out}**=**b_{in}** when $\varphi_1 = \varphi_2 = 0$, since $[BS_1]X[BS_2]=1$) :

$$\begin{pmatrix} a_{out} \\ b_{out} \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}}_{BS_2} \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}}_{BS_1} \begin{pmatrix} a_{in} \\ b_{in} \end{pmatrix} \qquad BS_1 \times BS_2 = 1$$

Optical interferometry (2)



Multiplying the matrices, and replacing the complex amplitudes by the corresponding photon annihilation operators, one gets:

$$\begin{pmatrix} \hat{a}_{out} \\ \hat{b}_{out} \end{pmatrix} = e^{i(\varphi_1+\varphi_2)/2} \begin{pmatrix} \cos(\varphi/2) & -\sin(\varphi/2) \\ \sin(\varphi/2) & \cos(\varphi/2) \end{pmatrix} \begin{pmatrix} \hat{a}_{in} \\ \hat{b}_{in} \end{pmatrix}, \qquad \varphi = \varphi_2 - \varphi_1,$$

where the operator â annihilates photons in mode a: $\hat{a}|N\rangle = \sqrt{N}|N-1\rangle$ and $|N\rangle$ is the Fock state with N photons, with $\hat{a}^\dagger\hat{a}|N\rangle = N|N\rangle$, where $\hat{a}^\dagger\hat{a}$ is the number operator. The overall phase above can be neglected.

We use now the Jordan-Schwinger transformation, which allows to analyze the Mach-Zender interferometer in terms of the algebra of angular momentum operators.

Optical interferometry and Jordan-Schwinger transformation

PHYSICAL REVIEW A VOLUME 33, NUMBER 6 JUNE 1986

SU(2) and SU(1,1) interferometers

Bernard Yurke, Samuel L. McCall, and John R. Klauder
AT&T Bell Laboratories, Murray Hill, New Jersey 07974

(Received 30 October 1985)

This has the advantage of providing a unified formalism, which can also be applied to problems in atomic spectroscopy and magnetometry.

Let $$\boxed{\hat{J}_x = \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}), \quad \hat{J}_y = \frac{i}{2}(\hat{b}^\dagger \hat{a} - \hat{a}^\dagger \hat{b}), \quad \hat{J}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})}$$

Then $[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k$ and $\hat{J}^2 = \frac{\hat{N}}{2}\left(\frac{\hat{N}}{2}+1\right), \quad \hat{N} = \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}$

so these operators obey the angular momentum algebra.

EXERCISE 3: Show this.

Transformations of operators $\hat{a}$ and $\hat{b}$ can be considered as rotations in angular momentum space: $\hat{a}_{out} = \hat{U}^\dagger \hat{a}_{in} \hat{U}$, $\hat{b}_{out} = \hat{U}^\dagger \hat{b}_{in} \hat{U}$, with $\hat{U} = \exp(-i\theta \hat{J} \cdot \hat{n})$, where the unit vector $\hat{n}$ is along the axis of rotation, with the correspondence:

BS1 $\rightarrow \hat{U} = \exp(-i\pi \hat{J}_x/2)$

BS2 $\rightarrow \hat{U} = \exp(i\pi \hat{J}_x/2)$

Phase delay $\rightarrow \hat{U} = \exp(-i\phi \hat{J}_z)$

$\phi = \varphi_2 - \varphi_1$

Angular momentum operators for optical interferometry

Corresponding transformation for the operators $\hat{J}_i$ (Heisenberg picture!):

$$\begin{pmatrix} \hat{J}_x^{out} \\ \hat{J}_y^{out} \\ \hat{J}_z^{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{J}_x^{in} \\ \hat{J}_y^{in} \\ \hat{J}_z^{in} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix} \begin{pmatrix} \hat{J}_x^{in} \\ \hat{J}_y^{in} \\ \hat{J}_z^{in} \end{pmatrix}$$

Therefore, Mach-Zender transformation amounts to a rotation around y axis of the angular momentum operators.

EXERCISE 4: Show this.

The state transforms as $$\boxed{|\psi\rangle_{out} = e^{i\hat{J}_x\pi/2} e^{-i\hat{J}_z\varphi} e^{-i\hat{J}_x\pi/2} |\psi\rangle_{in}}$$

Precision of phase estimation

From $\hat{J}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})$, it is clear that $\hat{n}_a - \hat{n}_b = 2\hat{J}_z$.

On the other hand, the average of $\hat{J}_z$ in the output state is equal to the average of $\hat{J}_z^{\text{out}}$, given by the previous matrix expression, in the input state.

Therefore, $\boxed{\langle \hat{J}_z \rangle_{\text{out}} = \cos\varphi \langle \hat{J}_z \rangle_{\text{in}} - \sin\varphi \langle \hat{J}_x \rangle_{\text{in}}}$ while the variance is

$$\boxed{\Delta^2 \hat{J}_z \Big|_{\text{out}} = \cos^2\varphi \, \Delta^2 \hat{J}_z \Big|_{\text{in}} + \sin^2\varphi \, \Delta^2 \hat{J}_x \Big|_{\text{in}} - 2\sin\varphi\cos\varphi \;\, \text{cov}(\hat{J}_x, \hat{J}_z)\Big|_{\text{in}}}$$

where the covariance cov is defined as

$$\boxed{\text{cov}(\hat{J}_x, \hat{J}_z) = \tfrac{1}{2}\langle \hat{J}_x \hat{J}_z + \hat{J}_z \hat{J}_x \rangle - \langle \hat{J}_x \rangle \langle \hat{J}_z \rangle}$$

EXERCISE 5:
Show this.

The precision of estimation can now be quantified by the error propagation formula:

$$\boxed{\Delta\varphi = \frac{\Delta \hat{J}_z \Big|_{\text{out}}}{\left| \frac{d\langle \hat{J}_z \rangle_{\text{out}}}{d\varphi} \right|}}$$

Phase is estimated from the difference in photon numbers at the two output doors

where $\Delta\varphi = \sqrt{\Delta^2\varphi}$ is a standard deviation (same for $\Delta\hat{J}_z$).

Optical interferometry with Fock states

Consider that a Fock state $|N\rangle$ is injected in port a, so that $\boxed{|\psi\rangle_{\rm in} = |N\rangle_a |0\rangle_b}$. Since

$$\hat{J}_x = \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}), \quad \hat{J}_y = \frac{i}{2}(\hat{b}^\dagger \hat{a} - \hat{a}^\dagger \hat{b}), \quad \hat{J}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) \quad \hat{J}^2 = (\hat{N}/2)(\hat{N}/2+1)$$

this initial state is an eingestate of $\hat{J}_z$ and $\hat{J}^2$: $\boxed{\hat{J}_z|N,0\rangle = (N/2)|N,0\rangle,}$ $\boxed{\hat{J}^2|N,0\rangle = \frac{N}{2}\left(\frac{N}{2}+1\right)|N,0\rangle,}$ so we may write $\boxed{|N,0\rangle \to |j,j\rangle\,.}$ Also,

$$\boxed{\langle \hat{J}_z\rangle_{\rm in} = N/2,\ \langle \hat{J}_x\rangle_{\rm in} = 0,\ \Delta^2 \hat{J}_z\Big|_{\rm in} = 0\,,\ \Delta^2 \hat{J}_x\Big|_{\rm in} = N/4,}$$

EXERCISE 6: Show this.

and $\boxed{{\rm cov}(\hat{J}_x, \hat{J}_z)_{\rm in} = 0\,.}$

From $\langle \hat{J}_z\rangle_{\rm out} = \cos\varphi \langle \hat{J}_z\rangle_{\rm in} - \sin\varphi \cancel{\langle \hat{J}_x\rangle_{\rm in}}$ and

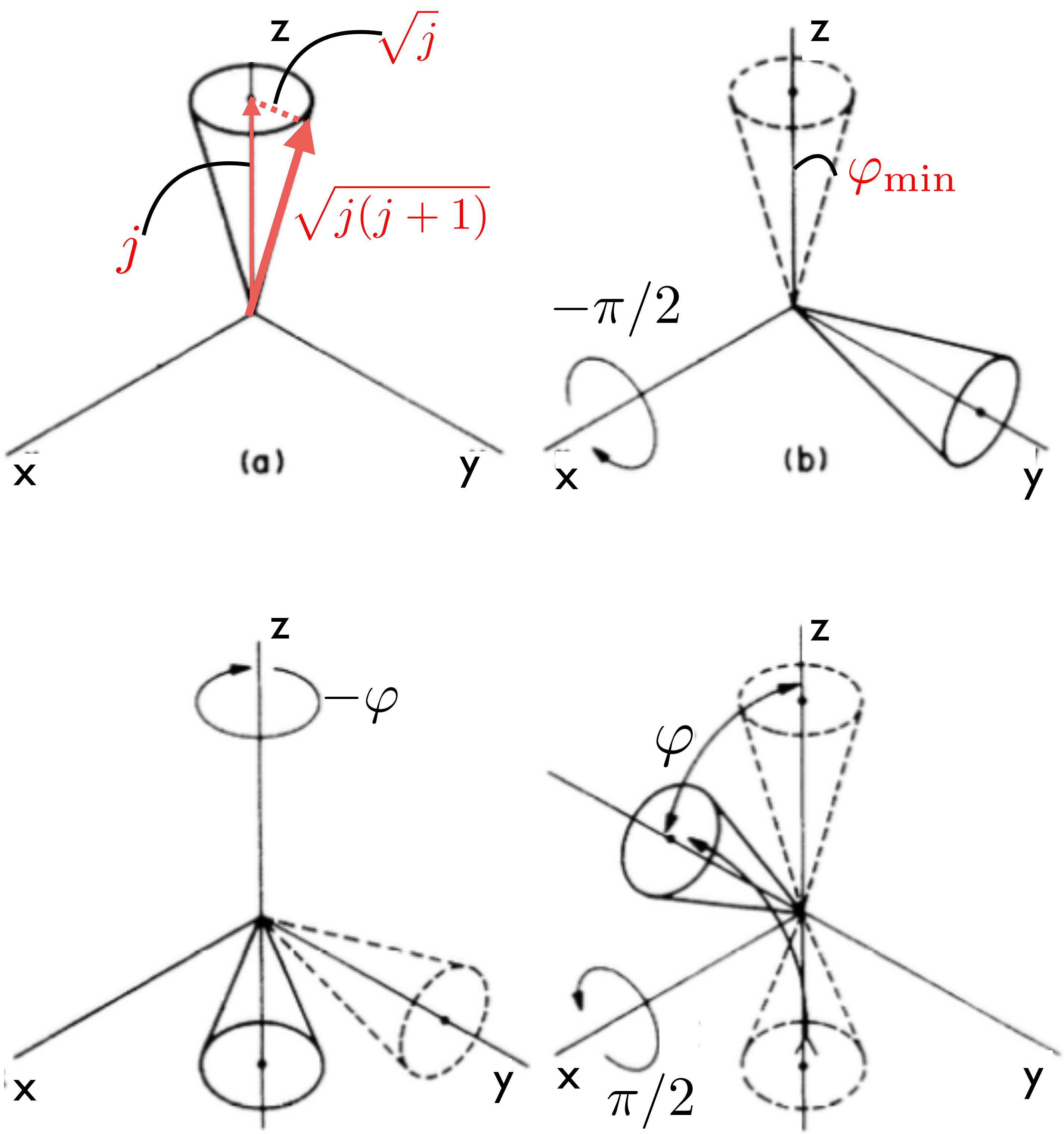
$$\Delta^2 \hat{J}_z\Big|_{\rm out} = \cos^2\varphi\ \cancel{\Delta^2 \hat{J}_z\Big|_{\rm in}} + \sin^2\varphi\ \Delta^2 \hat{J}_x\Big|_{\rm in} - 2\sin\varphi\cos\varphi\ \cancel{{\rm cov}(\hat{J}_x, \hat{J}_z)\Big|_{\rm in}}$$

one gets

$$\Delta\varphi = \frac{\Delta \hat{J}_z\Big|_{\rm out}}{\left|\frac{d\langle \hat{J}_z\rangle_{\rm out}}{d\varphi}\right|} = \frac{\sqrt{N}|\sin\varphi|/2}{N|\sin\varphi|/2} = \frac{1}{\sqrt{N}},$$

which is the standard (or shot-noise limit) for optical interferometry.

Geometrical interpretation



- Length of side of the cone: $\sqrt{j(j+1)}$, with j=N/2
- Distance from apex to center of base: eigenvalue of $\hat{J}_z$ —> j=N/2
- Radius of the base of the cone:
$$\sqrt{j(j+1) - j^2} = \sqrt{j}$$

(a) Initial state

(b) Action of first beam splitter

(c) Phase delay

(d) Action of second beam splitter

Minimum detectable φ is of the order of

$$\varphi_{\min} \approx \frac{\sqrt{j}}{j} = \frac{1}{\sqrt{j}} \approx \frac{1}{\sqrt{N}}$$

Optical interferometry with coherent states

Consider now that a coherent state $|\alpha\rangle$ is injected in port **a**, so that

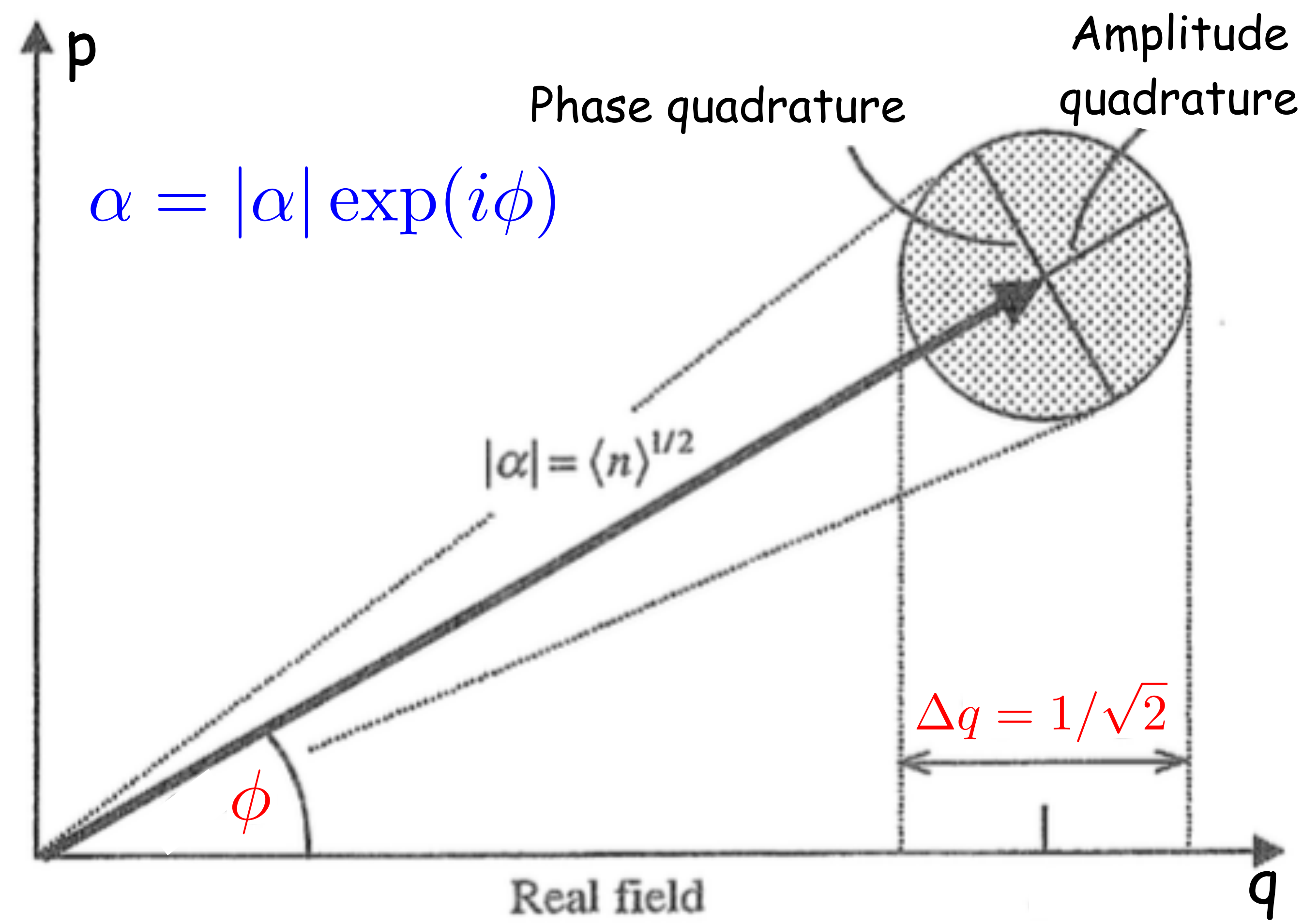
$$\boxed{|\psi\rangle_{\text{in}} = |\alpha\rangle_a |0\rangle_b}$$

Just to fix the notation (and also as a reminder...), a coherent state is an eigenstate of the operator â, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, and the average number of photons in the state is $\langle\alpha|\hat{N}|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2$.

Defining the quadrature operators as

$$\hat{q}_\theta = \frac{1}{\sqrt{2}}\left(\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}\right), \quad \hat{p}_\theta = \hat{q}_{\theta+\pi/2} = \frac{-i}{\sqrt{2}}\left(\hat{a}e^{-i\theta} - \hat{a}^\dagger e^{i\theta}\right), \text{with } [\hat{q}_\theta, \hat{p}_\theta] = i,$$

it follows that the corresponding standard deviations in the state $|\alpha\rangle$ are $\Delta p_\theta = \Delta q_\theta = 1/\sqrt{2}$, and the coherent state is a minimum uncertainty state: $\Delta p_\theta \Delta q_\theta = 1/2$



Optical interferometry with coherent states (2)

For the initial state $|\psi\rangle_{\text{in}} = |\alpha\rangle_a |0\rangle_b$, one has

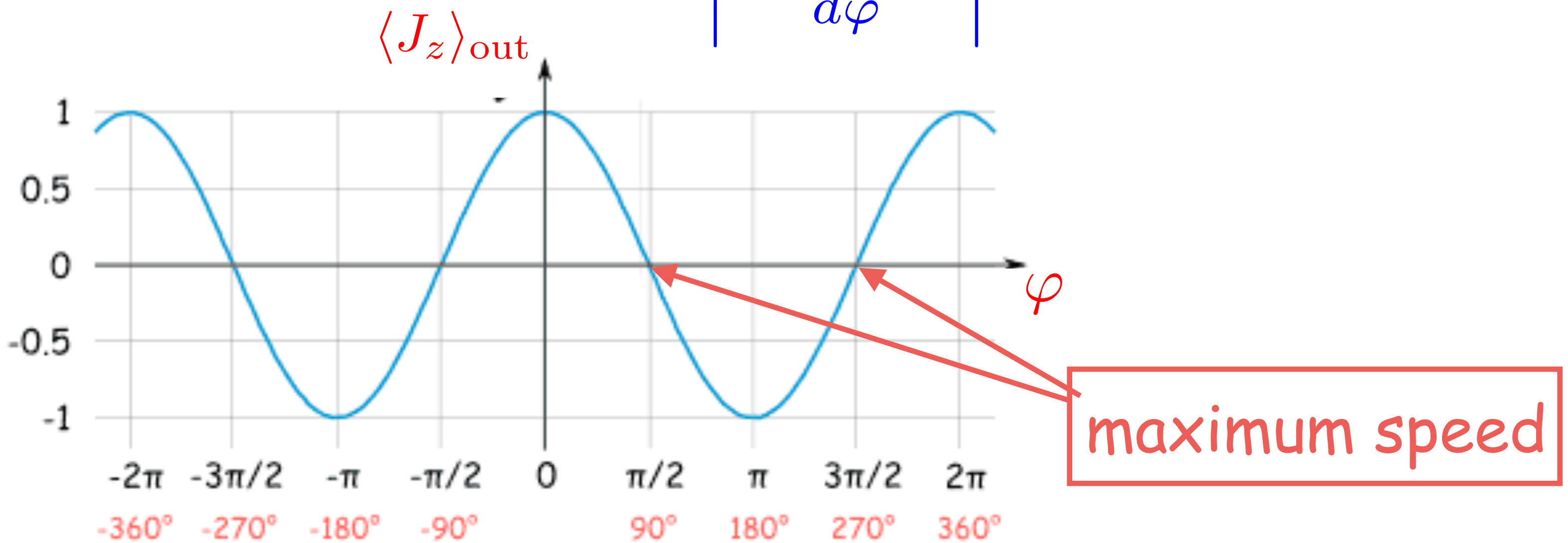
$$\boxed{\langle \hat{J}_z \rangle_{\text{in}} = \tfrac{1}{2}|\alpha|^2,\ \langle \hat{J}_x \rangle_{\text{in}} = 0,\ \Delta^2 \hat{J}_z\Big|_{\text{in}} = \Delta^2 \hat{J}_x\Big|_{\text{in}} = |\alpha|^2/4, \text{ and } \operatorname{cov}(\hat{J}_x, \hat{J}_z)_{\text{in}} = 0\,.}$$

From $\langle \hat{J}_z \rangle_{\text{out}} = \cos\varphi \langle \hat{J}_z \rangle_{\text{in}} - \sin\varphi \cancel{\langle \hat{J}_x \rangle_{\text{in}}}$ and

EXERCISE 7: Show this.

$$\Delta^2 \hat{J}_z\Big|_{\text{out}} = \cos^2\varphi\, \Delta^2 \hat{J}_z\Big|_{\text{in}} + \sin^2\varphi\, \Delta^2 \hat{J}_x\Big|_{\text{in}} - 2\sin\varphi\cos\varphi\ \cancel{\operatorname{cov}(\hat{J}_x, \hat{J}_z)\Big|_{\text{in}}}$$

one gets $$\Delta\varphi = \frac{\Delta \hat{J}_z\Big|_{\text{out}}}{\left|\frac{d\langle \hat{J}_z \rangle_{\text{out}}}{d\varphi}\right|} = \frac{|\alpha|/2}{|\alpha|^2|\sin\varphi|/2} = \frac{1}{|\alpha \sin\varphi|} = \frac{1}{\sqrt{\langle N \rangle}|\sin\varphi|}$$



Bound depends on incoming state and on the operating point!

The precision now depends on the operating point. The optimal operating points are at $\varphi = \pi/2$ or $\varphi = 3\pi/2$.

These two points correspond to the maximum speed of variation of $\langle \hat{J}_z \rangle_{\text{out}}$ with φ, implying higher sensitivity of $\langle \hat{J}_z \rangle_{\text{out}}$ to changes in this parameter.

Interferometry with coherent + squeezed states

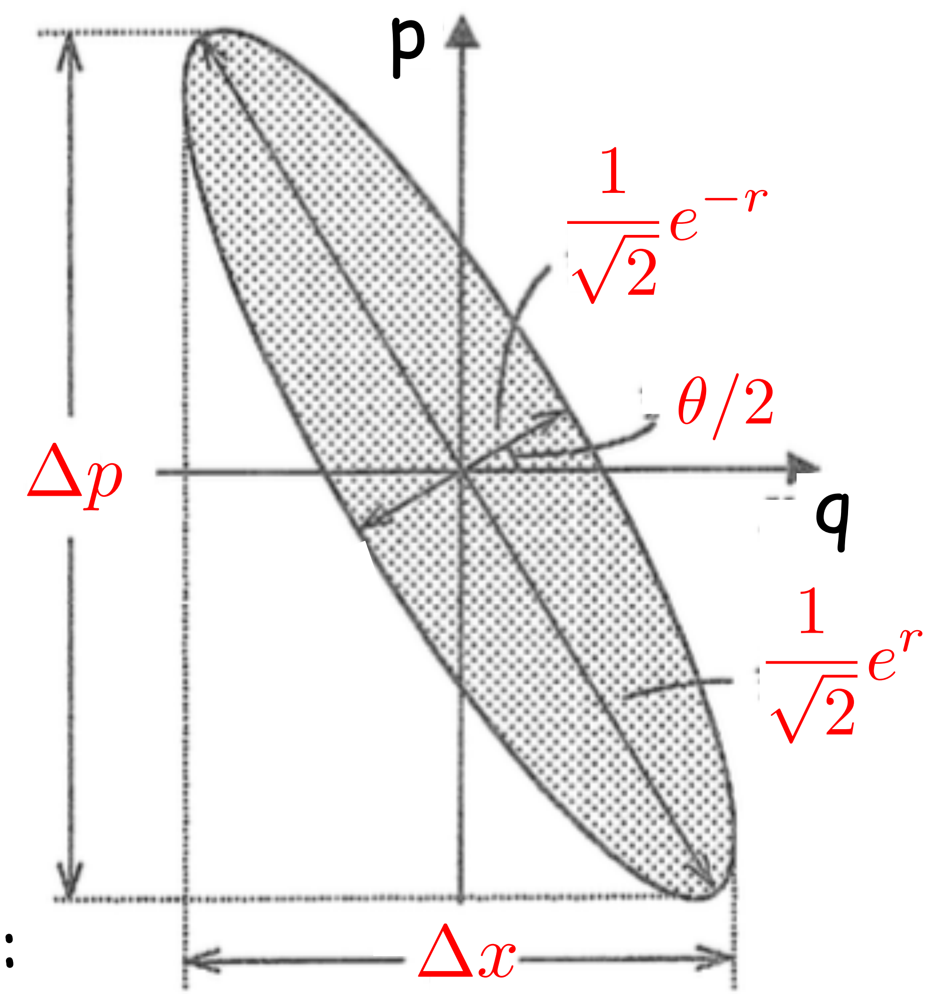
Important question: Can we do better, going beyond the shot-noise bound? This can actually be achieved, by using special quantum features of the incoming state.

Reminder on squeezed states

A squeezed state is a minimum-uncertainty state, obtained from a coherent state by a scaling transformation, which consists in squeezing a quadrature and stretching the orthogonal one. More formally, it is obtained from a coherent state through the transformation

$$\boxed{|\alpha, \xi\rangle = \hat{S}(\xi)|\alpha\rangle, \; \hat{S}(\xi) = \exp\left[\left(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}\right)/2\right]}$$

where $\xi = r\exp(i\theta)$ is an arbitrary complex number.



For $\xi = r$ real ($\theta = 0$), the uncertainties in q and p are:

$$\Delta q = e^{-r}/\sqrt{2}, \; \Delta p = e^{r}/\sqrt{2}$$

For metrology, the squeezed vacuum states are more relevant: $\boxed{|\xi\rangle = \hat{S}(\xi)|0\rangle.}$

The average number of photons in state $|\xi\rangle$ is $\langle\hat{N}\rangle = \sinh^2 r$: a squeezed vacuum state has an average number of photons different from zero.

Interferometry with coherent + squeezed states (2)

Assume now that a coherent state is injected into one of the ports of a Mach-Zender interferometer, and a vacuum squeezed state into the other port. The initial state is then $\boxed{|\alpha\rangle \otimes |\xi\rangle}$. This scheme was proposed by Caves in 1981, and is implemented in gravitational-wave interferometers (LIGO, GEO600).

Assuming for simplicity that $\xi = r$ is real (this fixes a direction in phase space), one has:

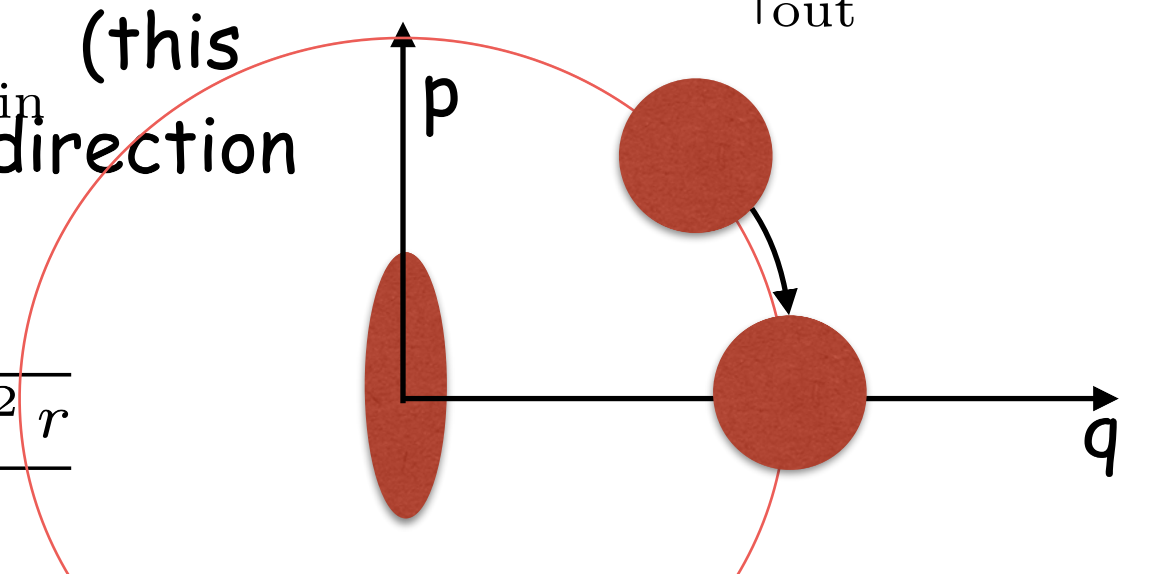
EXERCISE 8: Show this.

$$\langle \hat{N} \rangle = |\alpha|^2 + \sinh^2 r,\ \langle \hat{J}_z \rangle_{\text{in}} = (|\alpha|^2 - \sinh^2 r)/2,\ \langle \vec{J}_x \rangle_{\text{in}} = 0, \operatorname{cov}(\hat{J}_x, \hat{J}_z)\Big|_{\text{in}} = 0,$$

$$\Delta^2 \vec{J}_z\Big|_{\text{in}} = \left[|\alpha|^2 + (1/2)\sinh^2 2r\right]/4,\ \Delta^2 \vec{J}_x\Big|_{\text{in}} = \left[|\alpha|^2 \cosh 2r - \operatorname{Re}(\alpha^2)\sinh 2r + \sinh^2 r\right]/4\,.$$

This term reduces variance

Replacing these into the previous expressions for $\langle \hat{J}_z \rangle_{\text{out}}$ and $\Delta^2 \hat{J}_z\Big|_{\text{out}}$, and choosing α real, so as to minimize $\Delta^2 \hat{J}_x\Big|_{\text{in}}$ (this means that the coherent state is along the direction of highest compression):



$$\Delta\varphi = \frac{\sqrt{\cot^2\varphi(|\alpha|^2 + \frac{1}{2}\sinh^2 r) + |\alpha|^2 e^{-2r} + \sinh^2 r}}{\left||\alpha|^2 - \sinh^2 r\right|}$$

Interferometry with coherent + squeezed states (3)

We now try to minimize the expression:

$$\Delta\varphi = \frac{\sqrt{\cot^2\varphi(|\alpha|^2+\frac{1}{2}\sinh^2 r)+|\alpha|^2 e^{-2r}+\sinh^2 r}}{||\alpha|^2-\sinh^2 r|}$$

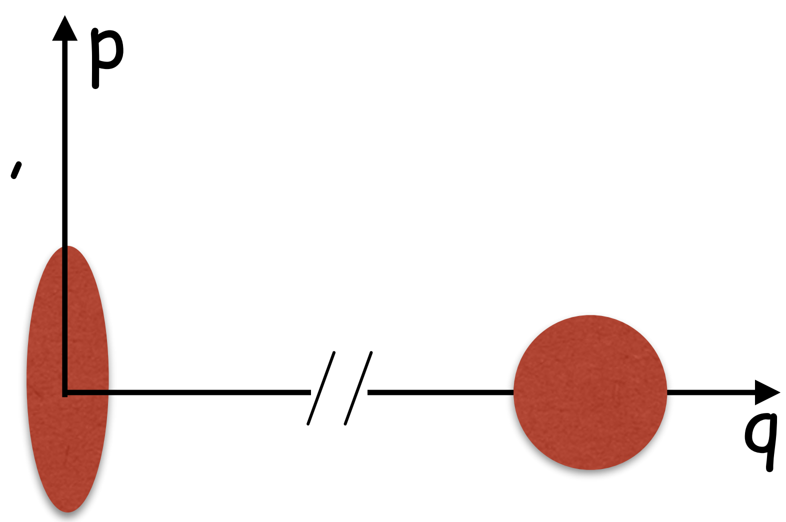
Optimal operation points: $\cot\varphi = 0 \Rightarrow \varphi = \pi/2,\ 3\pi/2.$

Then:

$$\Delta\varphi = \frac{\sqrt{|\alpha|^2 e^{-2r}+\sinh^2 r}}{||\alpha|^2-\sinh^2 r|}$$

Consider $\langle\hat{N}\rangle \gg 1$, with the squeezed vacuum carrying approximately $\sqrt{\langle\hat{N}\rangle}/2$ photons. Then the majority of photons belong to the coherent state, and $\sinh^2 r \approx (1/4)e^{2r} \approx \sqrt{\langle\hat{N}\rangle}/2$, so that

$$\lim_{\langle N\rangle\to\infty}\Delta\varphi \approx \frac{\sqrt{\langle N\rangle/(2\sqrt{N}) + \sqrt{\langle N\rangle}/2}}{\langle N\rangle - \sqrt{\langle N\rangle}/2} \approx \frac{1}{\langle N\rangle^{3/4}},$$

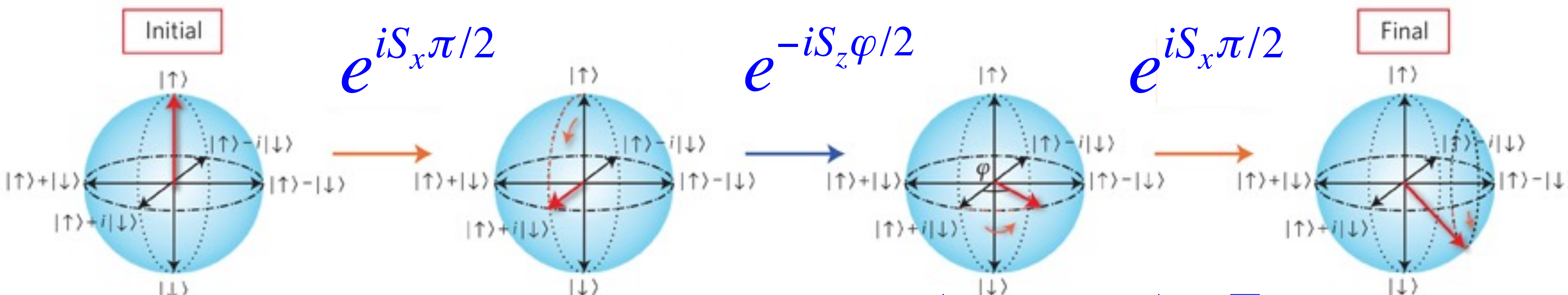


implying that this scheme leads to precision better than shot noise, for the same amount of resources — in this case, the average photon number $\langle N\rangle$.

Unified formalism for interferometers

Ramsey interferometry

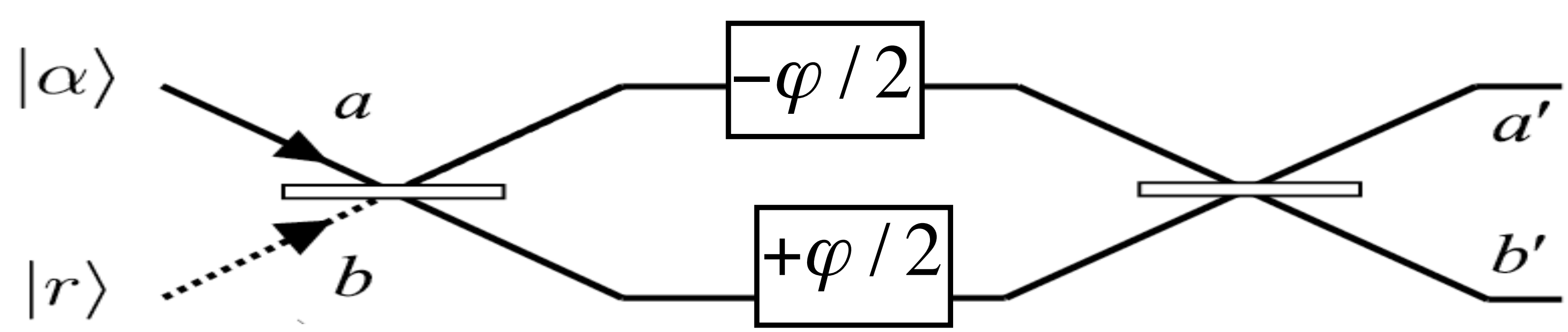
N independent atoms: Uncertaintiy in the phase scales as $1/\sqrt{N}$



$$|e\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|e\rangle \to \left(|e\rangle + i|g\rangle\right)/\sqrt{2} \to |e\rangle \to \left(|e\rangle + ie^{i\varphi}|g\rangle\right)/\sqrt{2} \to -|e\rangle\sin(\varphi/2) + |g\rangle\cos(\varphi/2)$$

$$\langle J_z \rangle = 2(P_g - P_e) = \cos\phi$$



$$|\psi\rangle_{\rm out} = e^{i\hat{J}_x\pi/2}e^{-i\hat{J}_z\varphi}e^{i\hat{J}_x\pi/2}|\psi\rangle_{\rm in}$$

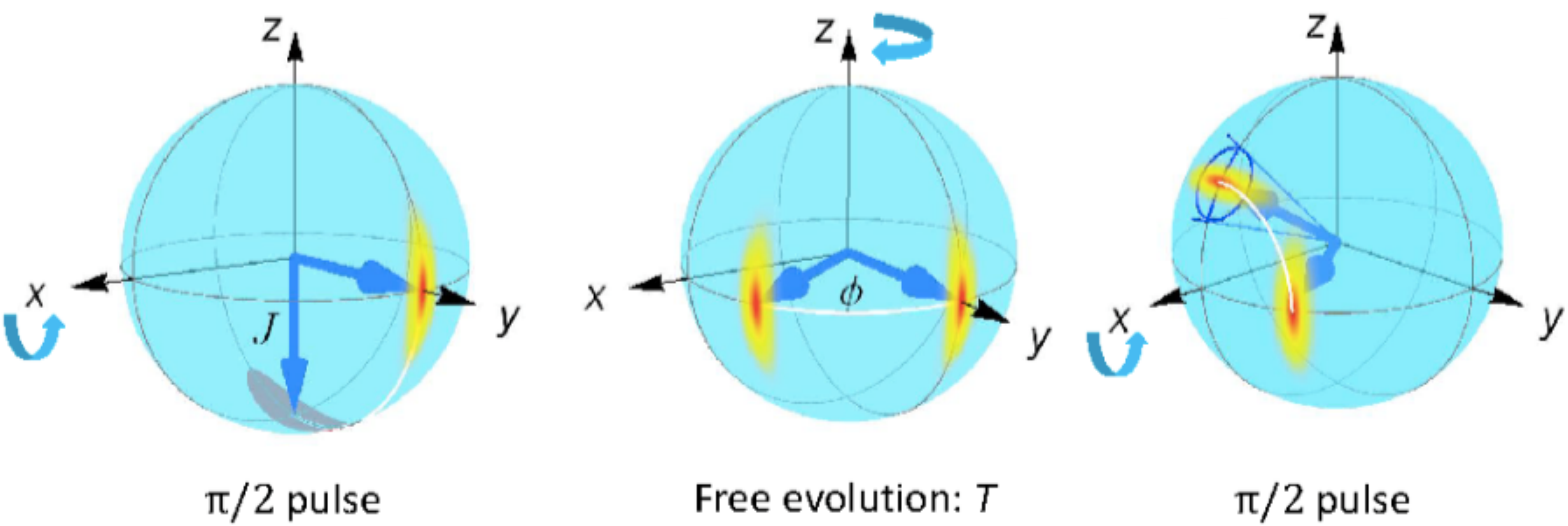
$$\varphi = \Delta\omega t$$

Getting better precision: Squeezed atomic states

It is also possible to prepare squeezed atomic states, which lead to a 1/N scaling. Starting with atoms in a ground state, squeezed atomic states are obtained through the transformation

$$|\psi_\xi\rangle = \exp[(-\xi/2)(J_+^2 - J_-^2)]|g\rangle^{\otimes N},\ \xi\ \text{real}$$

which is analogous to the corresponding transformation for electromagnetic fields. The successive transformations, applied on the collective angular momentum, are essentially the same as before — the squeezing reduces the final variance of J_z, thus increasing the precision in the estimation of the phase.



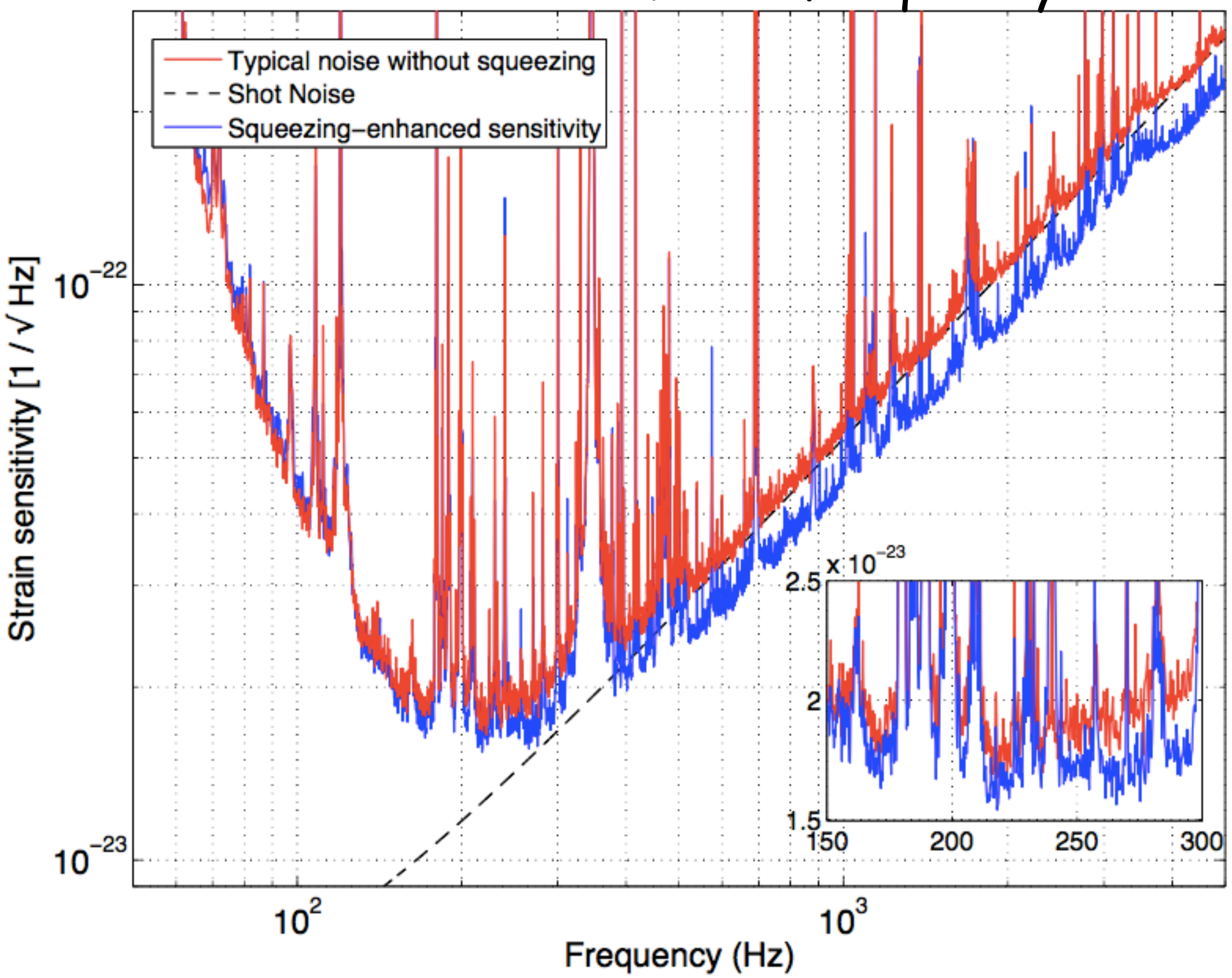
J. Ma, X. Wang, C. P. Sun, and F. Nori, arXiv:1011.2978 [quant-ph].

High-precision interferometry: Advanced LIGO

Differential displacement sensitivity $\simeq 10^{-19}\,\mathrm{m}$

Relative change in distance $\simeq 3\times 10^{-23}$

Up to 2.15 dB improvement in sensitivity in the shot-noise- limited frequency band

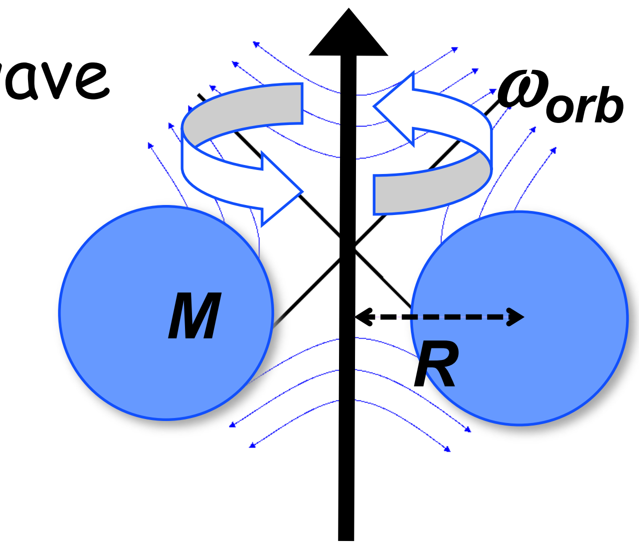


Gravitational-wave interferometer

Metric tensor $g_{\mu\nu}$ in general relativity (linearized): $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$\eta_{\mu\nu} \rightarrow$ Flat Minkowski space

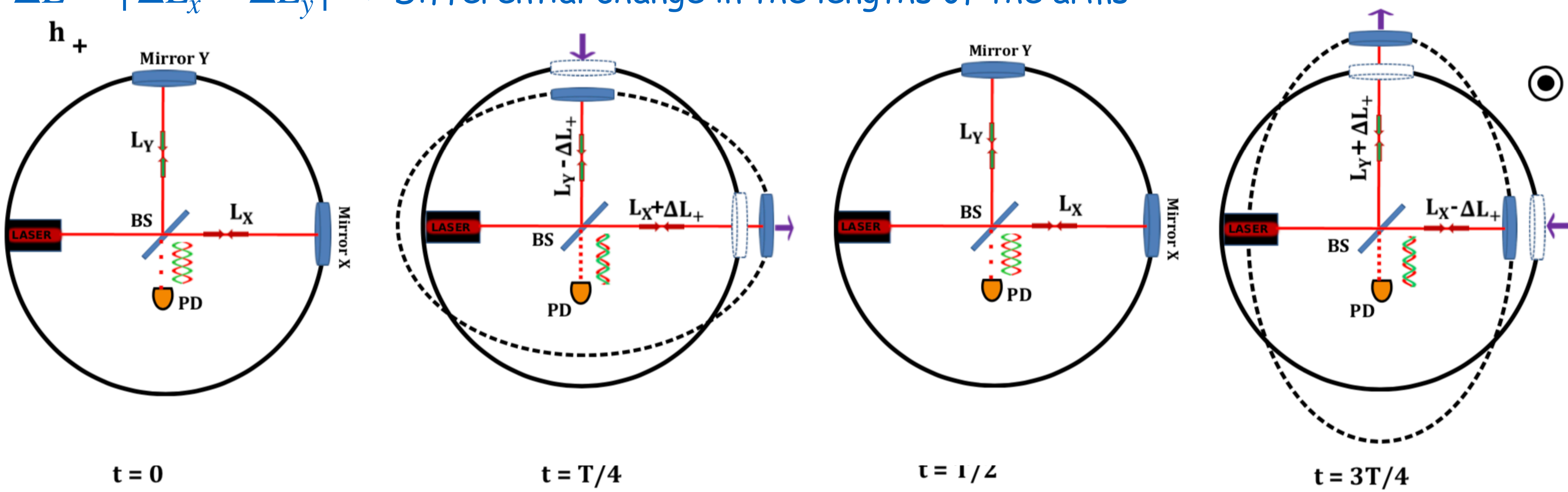
$h_{\mu\nu} \rightarrow$ Small perturbation representing the gravitational wave



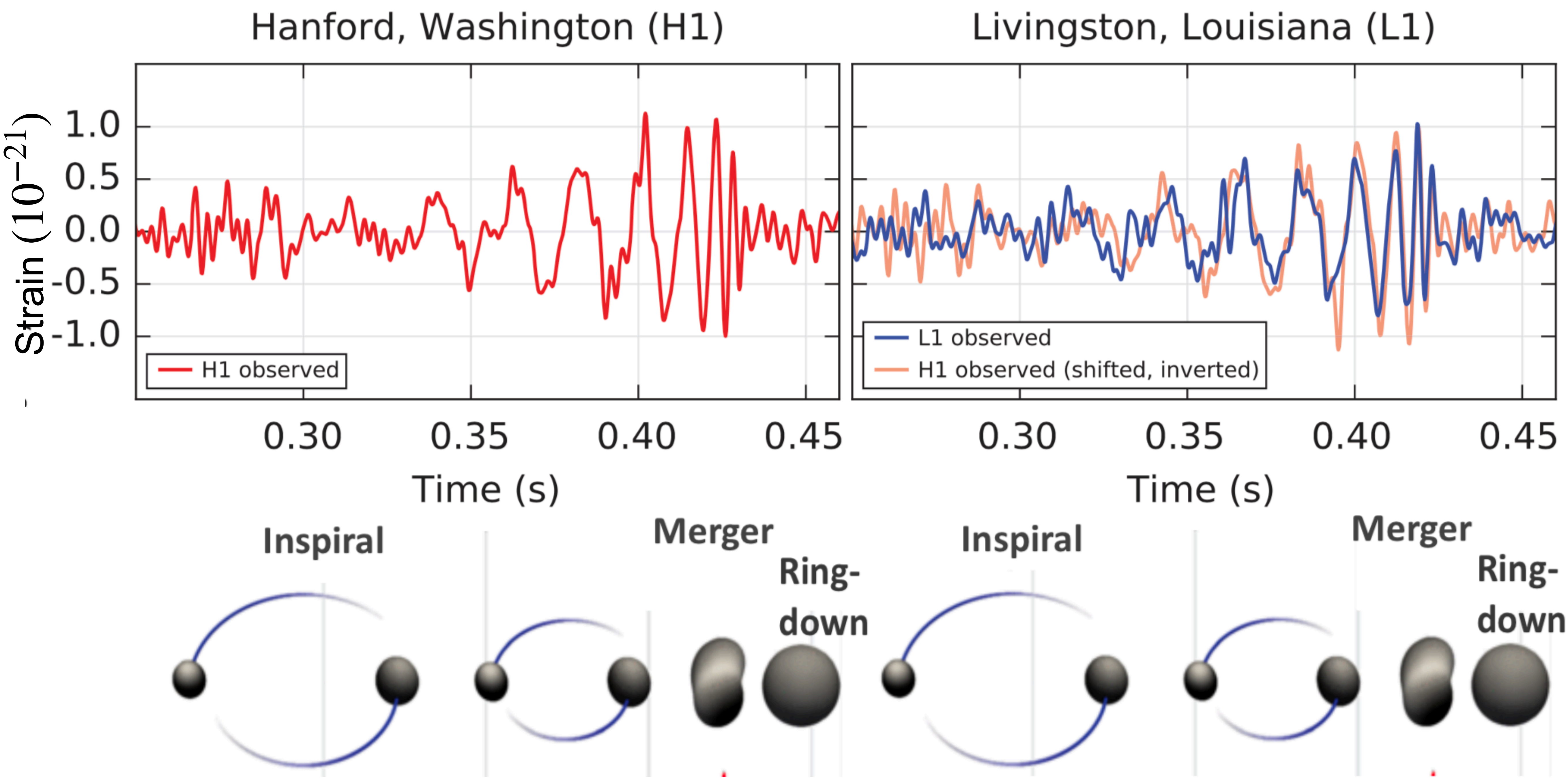
Strain amplitude: $h \approx 8GMR^2\omega_{orb}^2/rc^4 \sim 10^{-21} - 10^{-23}$
for binary black hole system (r=distance to observer)

Physically, *h* is a *strain*: ΔL/L

$\Delta L = |\Delta L_x - \Delta L_y| \rightarrow$ Differential change in the lengths of the arms

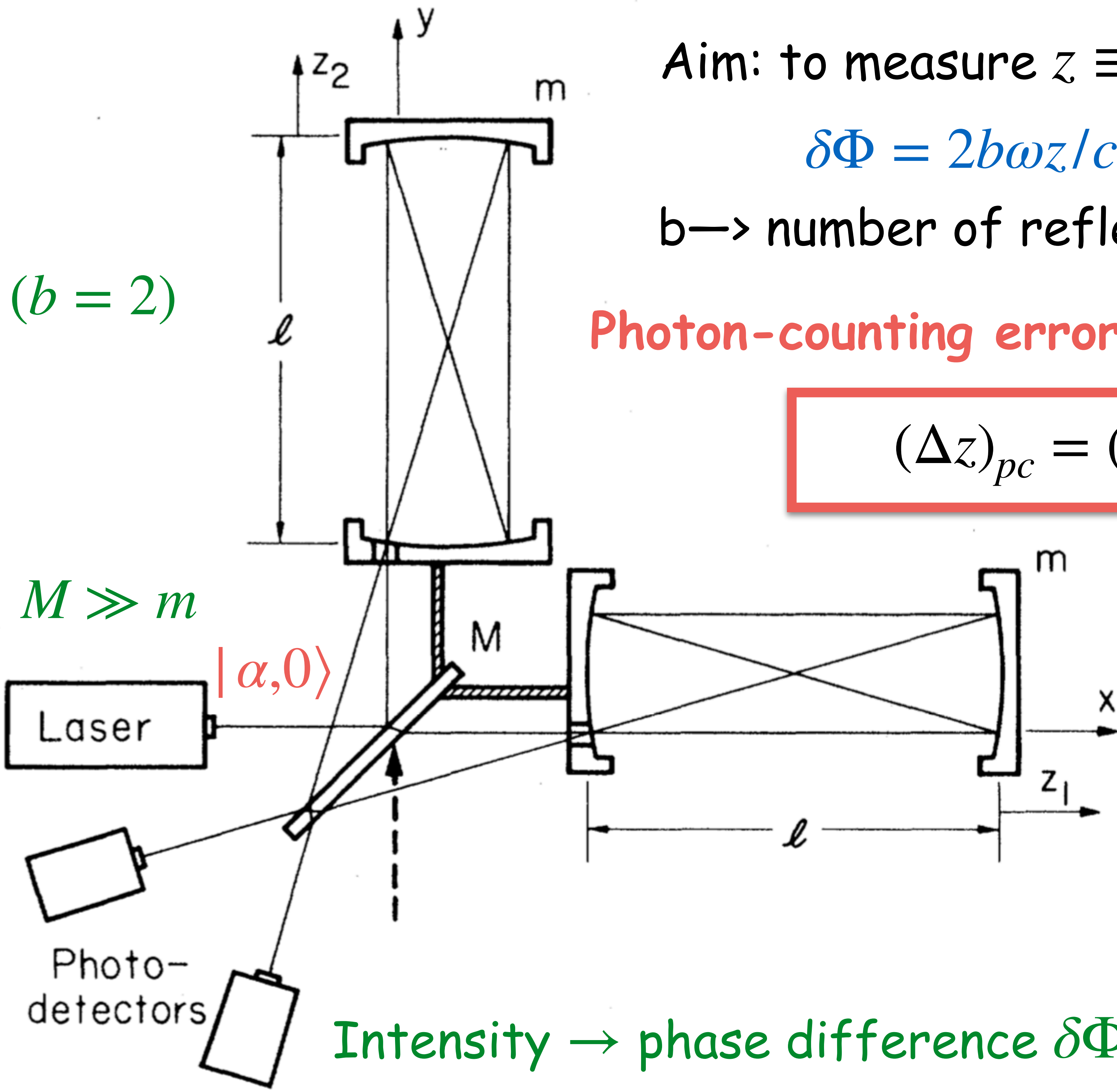


Gravitational-wave interferometer



Quantum standard limit in a gravitational interferometer

C. M. Caves, PRL **45**, 75 (1980)



$(b = 2)$

$M \gg m$

$|\alpha,0\rangle$

Aim: to measure $z \equiv z_2 - z_1$

$$\delta\Phi = 2b\omega z/c \Rightarrow z = (c/2b\omega)\delta\Phi$$

b—> number of reflections at each end mirror

Photon-counting error in z:

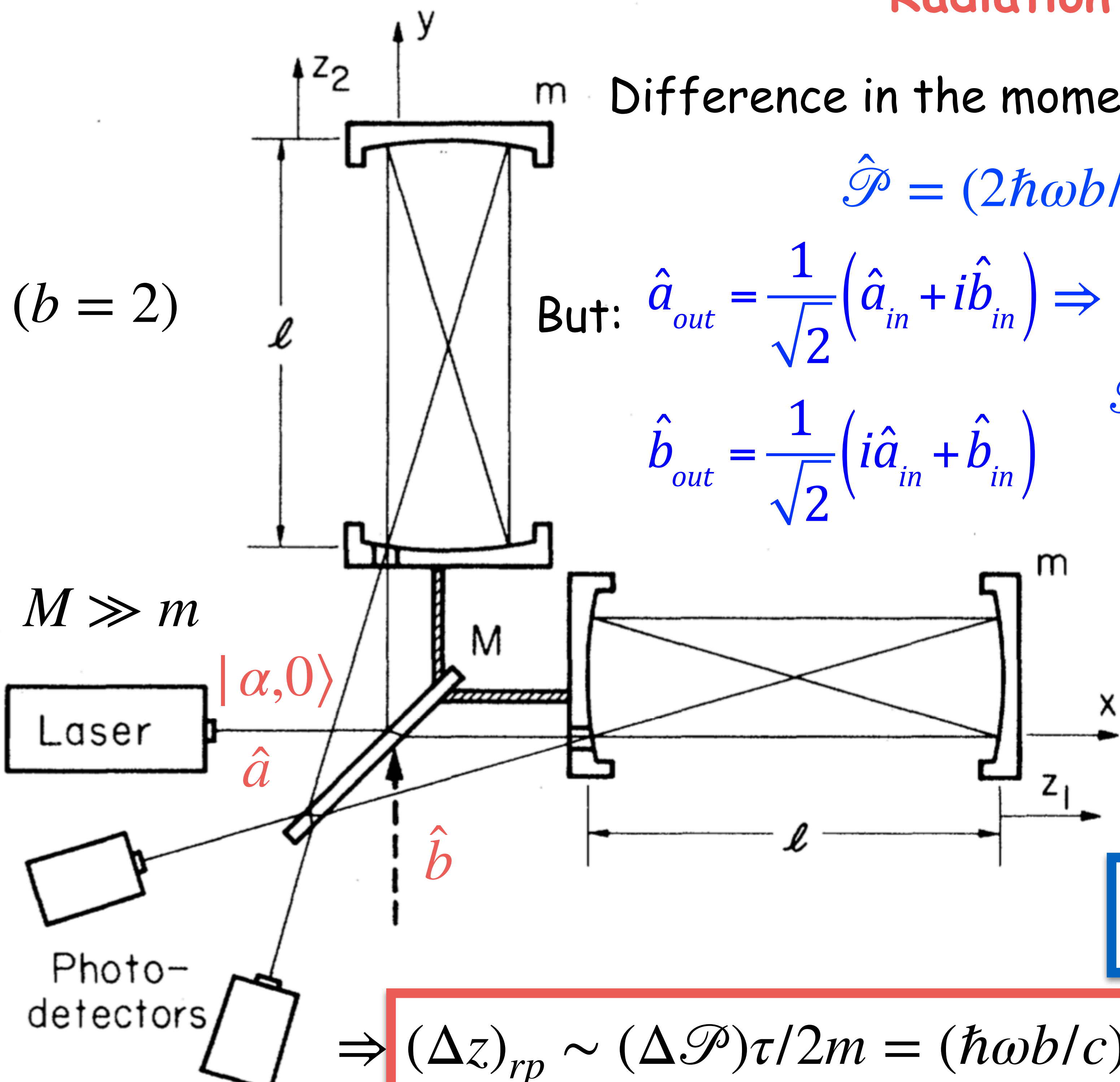
$$\boxed{(\Delta z)_{pc} = (c/2b\omega)\Delta(\delta\Phi) \sim (c/2b\omega)N^{-1/2}}$$

Minimum measuring time:
$\tau \sim 2b\ell/c \sim 1/f$, where f is the frequency of the gravitational wave. For $f = 100$ Hz, $\tau \sim 0.01$ seconds $\Rightarrow b\ell \sim 1500$ km

Intensity $\rightarrow$ phase difference $\delta\Phi$ between light in the two arms

Quantum standard limit in a gravitational interferometer

Radiation pressure



$(b = 2)$

$M \gg m$

$|\alpha,0\rangle$

$\hat{a}$

$\hat{b}$

Difference in the momenta transferred to the end masses:

$$\hat{\mathscr{P}} = (2\hbar\omega b/c)(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})_{\text{out}}$$

But: $\hat{a}_{out} = \dfrac{1}{\sqrt{2}}\left(\hat{a}_{in} + i\hat{b}_{in}\right) \Rightarrow \hat{\mathscr{P}} = i(2\hbar\omega b/c)(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})_{\text{in}}$

$$\hat{b}_{out} = \frac{1}{\sqrt{2}}\left(i\hat{a}_{in} + \hat{b}_{in}\right)$$

$$\hat{\mathscr{P}}^2 = (2\hbar\omega b/c^2)(\hat{a}^\dagger\hat{a}\hat{b}\hat{b}^\dagger + \hat{a}\cancel{\hat{a}^\dagger\hat{b}^\dagger}\hat{b} - \cancel{\hat{a}^{\dagger 2}}\hat{b}^2 - \cancel{\hat{b}^{\dagger 2}}\hat{a}^2)_{\text{in}}$$

Averages in state $|\alpha,0\rangle$:

$$\langle\hat{\mathscr{P}}\rangle = 0$$

$$\langle\hat{\mathscr{P}}^2\rangle = (2\hbar\omega b/c)^2|\alpha|^2$$

$$\boxed{\Delta\mathscr{P} = \langle\hat{\mathscr{P}}^2\rangle^{1/2} = (2\hbar\omega b/c)N^{1/2}}$$

$$\Rightarrow \boxed{(\Delta z)_{rp} \sim (\Delta\mathscr{P})\tau/2m = (\hbar\omega b/c)(\tau/m)N^{1/2}}$$

$\tau \rightarrow$ measuring time

Quantum standard limit in a gravitational interferometer

Photon-counting

$$\boxed{(\Delta z)_{pc} = (c/2b\omega)\Delta(\delta\Phi) \sim (c/2b\omega)N^{-1/2} = (c/2b)(\hbar/P\omega\tau)^{1/2} \sim (c/2b)(\hbar f/P\omega)^{1/2}}$$

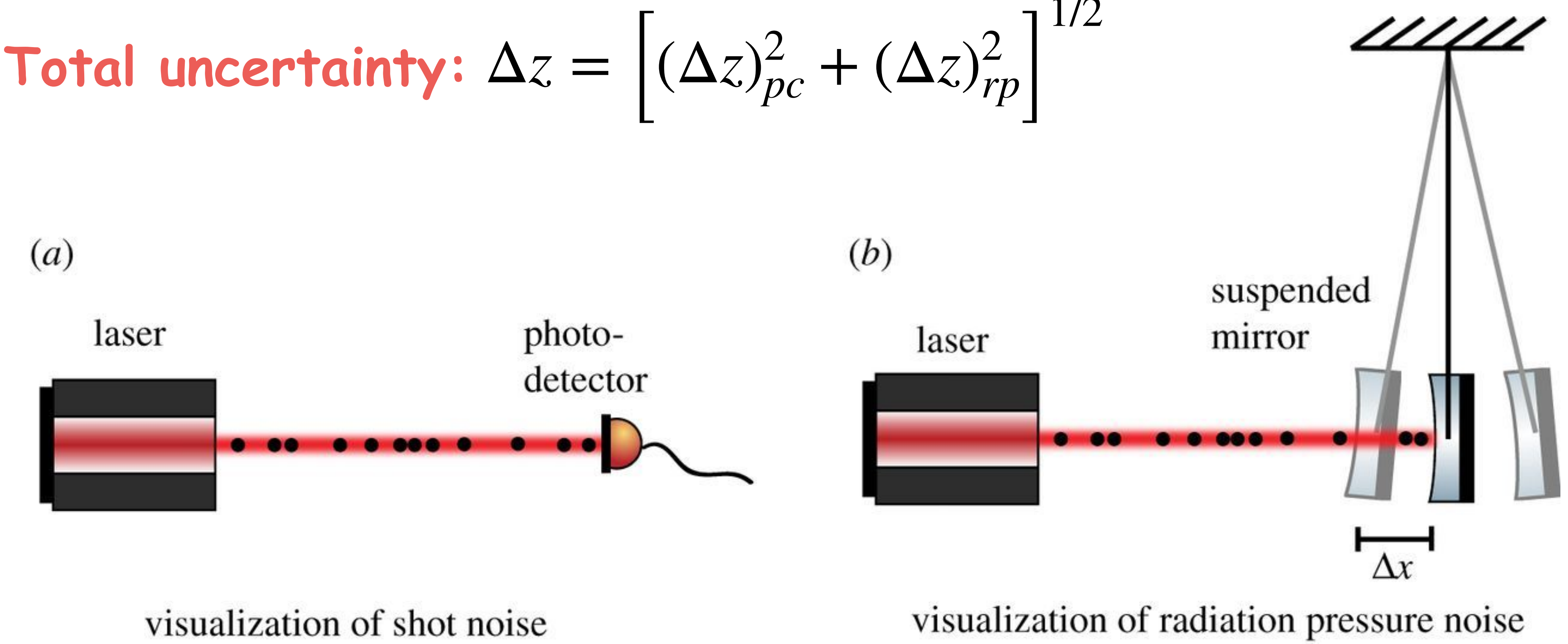
Radiation pressure

$$\boxed{(\Delta z)_{rp} \sim (\Delta\mathscr{P})\tau/2m = (\hbar\omega b/c)(\tau/m)N^{1/2} = (b/mc)(\hbar\omega P\tau^3)^{1/2} \sim (b/mc)(\hbar\omega P/f^3)^{1/2}}$$

where N was expressed in terms of the power $P = N\hbar\omega/\tau$, and τ was replaced by $1/f$, with f being the frequency of the gravitational wave.

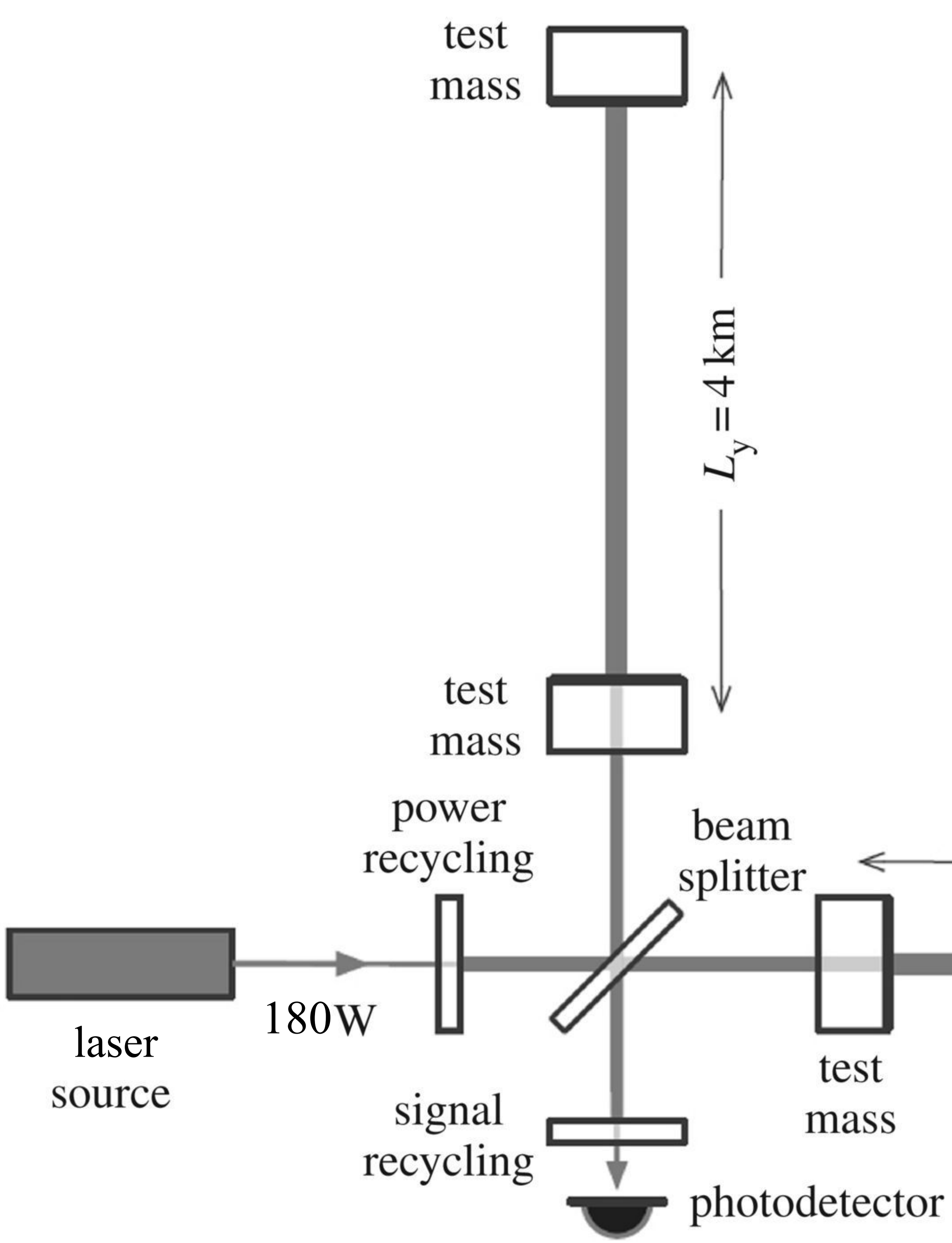
More important for low frequencies

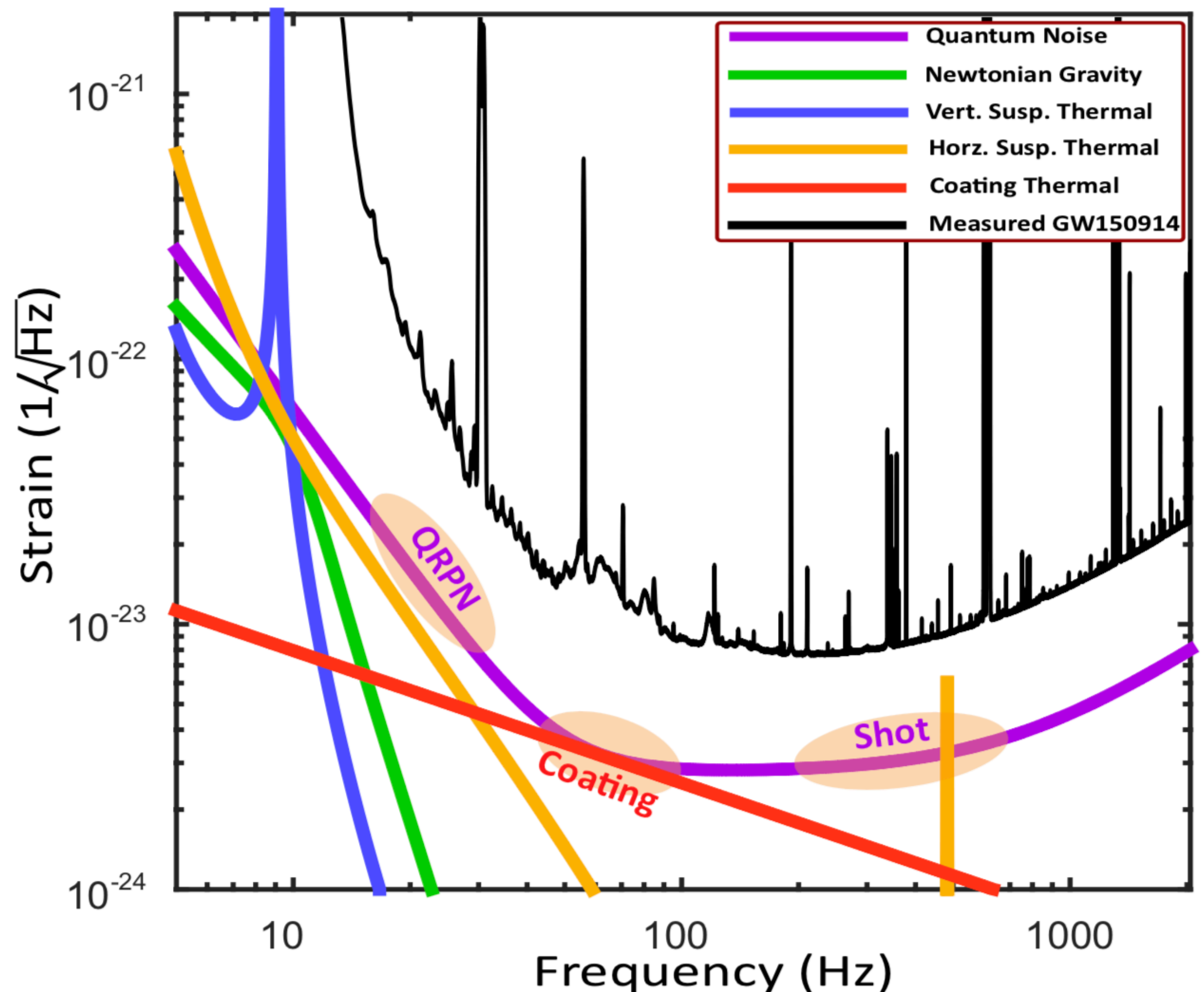
Total uncertainty: $\Delta z = \left[(\Delta z)^2_{pc} + (\Delta z)^2_{rp}\right]^{1/2}$



Quantum standard limit in a gravitational interferometer

Physically, *h* is a *strain*: ΔL/L





Increase in optical path length: about 300

General estimation theory

We have shown that it is possible to win over the shot noise in optical interferometry, by using states with specific quantum features, like states with well-defined number of photons or squeezed states. In these examples, the estimation was obtained through measurement of the difference of photon numbers in the outgoing arms of the interferometer. It is not clear whether these are the best possible measurements, or whether better bounds can be obtained by using other incoming states.

One may ask whether it is possible to find general bounds and strategies for reaching them, which could be applied to many different systems, and could eventually help us to identify which are the best states and the best measurements for achieving the best possible precision.

This is the aim of this series of lectures: to develop, and apply to examples, a general estimation theory, capable not only to consider unitary evolutions of closed systems, like the one described here for the optical interferometer, but also open (noisy) systems.

General estimation theory

1. What are the best possible measurements?

2. What are the best incoming states, in order to get better precision?

3. Is it possible to find *general* bounds and strategies for reaching them, which could be applied to many different systems?